26. FUNDAMENTAL CONCEPTS OF 3-DIMENSIONAL GEOMETRY

This chapter consists of some important concepts of three-dimensional geometry. Though we have studied these in Class 11, yet we shall review here these fundamental concepts for ready reference.

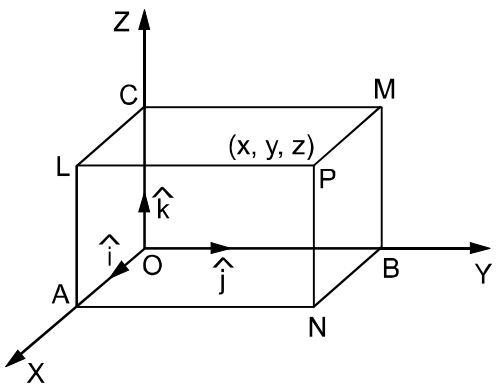
Coordinates of a Point in Space

Let O be the origin, and let OX, OY and OZ be three mutually perpendicular lines, taken as the x-axis, y-axis and z-axis respectively in such a way that they form a right-handed system.

The planes YOZ, ZOX and XOY are respectively known as the yz-plane, the zx-plane and the xy-plane.

These planes, known as the coordinate planes, divide the space into eight parts called *octants.*

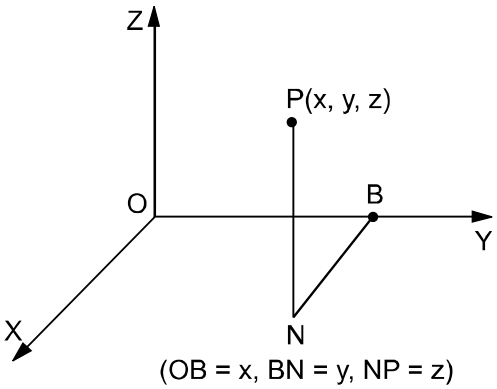
Let P be a point in space. Through P, draw three planes $PLAN$, $PNBM$ and $PLCM$ parallel to the yz-plane, the zx-plane and the xy-plane respectively, and meeting the x-axis, y-axis and z-axis at the points A, B, C respectively. Complete the parallelepiped whose coterminous edges are OA, OB and OC.



Let $OA = x$, $OB = y$ and $OC = z$. We say that the coordinates of P are (x, y, z).

It is clear from the figure alongside that

(i) x = distance of P from the yz-plane
(ii) y = distance of P from the zx-plane
(iii) z = distance of P from the xy-plane.



Also, we can say that

(i) the equation of the xy-plane is $z = 0$
(ii) the equation of the xz-plane is $y = 0$
(iii) the equation of the yz-plane is $x = 0$.

POSITION VECTOR OF A POINT IN SPACE Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along OX, OY and OZ respectively.

If P(a, b, c) is a point in space, we say that the position vector (or, *p.v.) of P is* $(a\hat{i} + b\hat{j} + c\hat{k})$.

Some Results on Points in Space

1. DISTANCE BETWEEN TWO POINTS The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

2. SECTION FORMULAE

(i) If $P(x, y, z)$ divides the join of $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$ then

$$x = \frac{(mx_2 + nx_1)}{(m+n)}, \quad y = \frac{(my_2 + ny_1)}{(m+n)}, \quad z = \frac{(mz_2 + nz_1)}{(m+n)}.$$

(ii) The mid-point of the line joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

(iii) The centroid of $\triangle ABC$ with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

Some Results on Lines in Space

1. DIRECTION COSINES OF A LINE If a line makes angles α, β, γ with the x-axis, y-axis and z-axis respectively then

$$l = \cos\alpha, m = \cos\beta, n = \cos\gamma$$

are called the direction cosines (or, d.c.'s) of the line.

We always have $l^2 + m^2 + n^2 = 1$.

REMARKS (i) d.c.'s of the x-axis are 1, 0, 0.
(ii) d.c.'s of the y-axis are 0, 1, 0.
(iii) d.c.'s of the z-axis are 0, 0, 1.

2. DIRECTION RATIOS OF A LINE Any three numbers a, b, c, proportional to the direction cosines l, m, n respectively of a line, are called the direction ratios of the line.

Clearly, we have $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

Some Important Facts

(i) If a, b, c are the direction ratios of a line then its direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

(ii) If $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ then the direction ratios of $\vec{r}$ are a, b, c.

(iii) Let PQ be a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Then, the direction ratios of the line PQ are $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$.

3. ANGLE BETWEEN TWO LINES If θ is the angle between two lines L_1 and L_2 whose d.c.'s are l_1, m_1, n_1, and l_2, m_2, n_2 then the following hold true.

(i) $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

(ii) $\sin\theta = \sqrt{\Sigma(m_1n_2 - m_2n_1)^2}$

(iii) lines L_1 and L_2 are perpendicular $\Leftrightarrow$ $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(iv) lines L_1 and L_2 are parallel $\Leftrightarrow$ $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

REMARKS In fact, there are two angles θ and $(\pi - \theta)$ between two lines.

SOLVED EXAMPLES

EXAMPLE 1 *Find the direction cosines of a line whose direction ratios are* $2, -6, 3$.

SOLUTION Here, $a = 2, b = -6, c = 3$.

$$\therefore \quad \sqrt{a^2 + b^2 + c^2} = \sqrt{2^2 + (-6)^2 + 3^2}$$

$$= \sqrt{4 + 36 + 9} = \sqrt{49} = 7.$$

Hence, the direction cosines of the given line are $\frac{2}{7}, \frac{-6}{7}, \frac{3}{7}$.

EXAMPLE 2 *Find the direction cosines of each of the following vectors:*

(i) $2\hat{i} + \hat{j} - 2\hat{k}$ *(ii)* $-\hat{i} - \hat{k}$ *(iii)* $-\hat{j}$

SOLUTION (i) Direction ratios of the vector $(2\hat{i} + \hat{j} - 2\hat{k})$ are 2, 1, –2.

And, $\sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$.

$\therefore$ d.c.'s of the given vector are $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$.

(ii) Direction ratios of the vector $(-\hat{i} - \hat{k})$ are –1, 0, –1.

And, $\sqrt{(-1)^2 + 0^2 + (-1)^2} = \sqrt{1 + 0 + 1} = \sqrt{2}$.

$\therefore$ d.c.'s of the given vector are $\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$.

(iii) Direction ratios of the vector $-\hat{j}$ are 0, –1, 0.

And, $\sqrt{0^2 + (-1)^2 + 0^2} = \sqrt{1} = 1$.

$\therefore$ d.c.'s of the given vector are 0, –1, 0.

EXAMPLE 3 *Find the direction cosines of a line which makes angles* $90°, 135°$ *and* $45°$ *with the positive directions of the x-, y-, and z-axis respectively.*

SOLUTION The direction cosines of the given line are

$$\cos 90°, \cos 135°, \cos 45°, \text{ i.e., } 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}.$$

EXAMPLE 4 *If a line makes angles* α, β *and* γ *with the coordinate axes, prove that* $(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 2.$

SOLUTION Let the direction cosines of the given line be *l, m, n*. Then,

$$l = \cos\alpha, \quad m = \cos\beta \text{ and } n = \cos\gamma.$$

$$\therefore \quad (l^2 + m^2 + n^2) = 1 \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

Hence, $(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 2.$

EXAMPLE 5 *If a line makes angles* α, β *and* γ *with the coordinate axes, prove that* $(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = -1.$

SOLUTION Let the direction cosines of the given line be *l, m, n*. Then,

$$l = \cos\alpha, \quad m = \cos\beta \text{ and } n = \cos\gamma.$$

$$\therefore \quad (l^2 + m^2 + n^2) = 1 \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow 2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma = 2$$

$$\Rightarrow (1 + \cos 2\alpha) + (1 + \cos 2\beta) + (1 + \cos 2\gamma) = 2$$

$$\Rightarrow (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = -1.$$

Hence, $(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = -1.$

EXAMPLE 6 *Find the direction cosines of a line that makes equal angles with the coordinate axes.* **[CBSE 2011]**

SOLUTION Let the given line make angle α with each of the coordinate axes and let the direction cosines of this line be *l, m, n*. Then,

$$l = m = n = \cos\alpha.$$

$$\therefore \quad (l^2 + m^2 + n^2) = 1 \Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow 3\cos^2\alpha = 1 \Rightarrow \cos^2 = \frac{1}{3}$$

$$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}.$$

Hence, the d.c.'s of the given line are

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \text{ or } \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}.$$

EXAMPLE 7 *A line makes angle* $60°$ *and* $45°$ *with the positive direction of x-axis and y-axis respectively. What acute angle does it make with the z-axis?*

SOLUTION Let the given line make an angle γ with the z-axis. Then,

$$\cos^2 60° + \cos^2 45° + \cos^2 \gamma = 1$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2 \gamma = 1$$

$$\Rightarrow \quad \left(\frac{1}{4} + \frac{1}{2}\right) + \cos^2 \gamma = 1 \;\Rightarrow\; \cos^2 \gamma = \left(1 - \frac{3}{4}\right) = \frac{1}{4}$$

$$\Rightarrow \quad \cos \gamma = \pm \frac{1}{2}$$

$$\Rightarrow \quad \gamma = 60° \quad \text{or} \quad 120°.$$

Hence, the acute angle which the given line makes with the z-axis is $60°$.

EXAMPLE 8 *Find the direction cosines of the vector* $\vec{r} = (6\hat{i} + 2\hat{j} - 3\hat{k})$.

SOLUTION The direction ratios of $\vec{r}$ are 6, 2, –3.

And, $\sqrt{6^2 + 2^2 + (-3)^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7$.

Hence, the direction cosines of $\vec{r}$ are $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$.

EXAMPLE 9 *Find the direction cosines of the line segment joining the points* $A(7, -5, 9)$ *and* $B(5, -3, 8)$.

SOLUTION D.r.'s of AB are $(5-7), (-3+5), (8-9)$, i.e., –2, 2, –1.

$$|AB| = \sqrt{(-2)^2 + 2^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3.$$

Hence, the d.c.'s of the line AB are $\frac{-2}{3}, \frac{2}{3}, \frac{-1}{3}$.

EXAMPLE 10 *Find the angles made by the vector* $\vec{r} = (\hat{i} + \hat{j} - \hat{k})$ *with the coordinate axes.*

SOLUTION D.r.'s of the given vector $\vec{r}$ are 1, 1, –1.

$$|\vec{r}| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}.$$

$\therefore$ d.c.'s of $\vec{r}$ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$.

Let $\vec{r}$ make angles α, β and γ with the x-axis, y-axis and z-axis respectively. Then, d.c.'s of the x-axis are 1, 0, 0.

$$\therefore \quad \cos\alpha = \left\{\left(\frac{1}{\sqrt{3}}\times 1\right)+\left(\frac{1}{\sqrt{3}}\times 0\right)+\left(\frac{1}{\sqrt{3}}\times 0\right)\right\} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Similarly, $\cos\beta = \frac{1}{\sqrt{3}} \Rightarrow \beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

And, $\cos\gamma = \frac{-1}{\sqrt{3}} \Rightarrow \gamma = \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$.

Hence, the angles made by the given vector $\vec{r}$ with the x-axis, y-axis and z-axis are $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ respectively.

EXAMPLE 11 *Find the angle between the lines whose direction ratios are* $3, 2, -6$ *and* $1, 2, 2$.

SOLUTION Let the given lines be L_1 and L_2 respectively. Then,

D.r.'s of L_1 are $3, 2, -6$.

And, $\sqrt{3^2+2^2+(-6)^2} = \sqrt{9+4+36} = \sqrt{49} = 7$.

$\therefore$ d.c.'s of L_1 are $\frac{3}{7}, \frac{2}{7}, \frac{-6}{7}$.

D.r.'s of L_2 are $1, 2, 2$.

And, $\sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$.

$\therefore$ d.c.'s of L_2 are $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$.

Let θ be the angle between L_1 and L_2. Then,

$$\cos\theta = |\, l_1l_2 + m_1m_2 + n_1n_2\,|$$

$$= \left|\left(\frac{3}{7}\times\frac{1}{3}\right)+\left(\frac{2}{7}\times\frac{2}{3}\right)+\left(\frac{-6}{7}\times\frac{2}{3}\right)\right|$$

$$= \left|\frac{1}{7}+\frac{4}{21}-\frac{4}{7}\right| = \left|\frac{-5}{21}\right| = \frac{5}{21}.$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{5}{21}\right).$$

EXAMPLE 12 *Find the angle between the vectors* $\vec{r}_1 = (4\hat{i} - 3\hat{j} + 5\hat{k})$ *and* $\vec{r}_2 = (3\hat{i} + 4\hat{j} + 5\hat{k})$.

SOLUTION Let θ be the angle between $\vec{r}_1$ and $\vec{r}_2$. Then,

$$\cos\theta = \frac{\vec{r}_1\cdot\vec{r}_2}{|\vec{r}_1||\vec{r}_2|} = \frac{(4\hat{i}-3\hat{j}+5\hat{k})\cdot(3\hat{i}+4\hat{j}+5\hat{k})}{\{\sqrt{4^2+(-3)^2+5^2}\}\{\sqrt{3^2+4^2+5^2}\}}$$

$$= \frac{(12-12+25)}{\{\sqrt{16+9+25}\}\{\sqrt{9+16+25}\}} = \frac{25}{\{\sqrt{50}\times\sqrt{50}\}} = \frac{25}{50} = \frac{1}{2}.$$

$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$

Hence, the angle between the given vectors is $\frac{\pi}{3}$.

EXAMPLE 13 *Find the direction cosines of the line which is perpendicular to each of the lines having direction ratios* $1, -2, -2$ *and* $0, 2, 1$ *respectively.*

SOLUTION Let the direction ratios of the required line be a, b, c. Then, it being perpendicular to each of the lines having d.r.'s 1, –2, –2 and 0, 2, 1 we have

$$a - 2b - 2c = 0 \qquad \text{... (i)}$$

$$0a + 2b + c = 0. \qquad \text{... (ii)}$$

On solving (i) and (ii) by cross multiplication, we have

$$\frac{a}{(-2+4)} = \frac{b}{(0-1)} = \frac{c}{(2-0)} \Rightarrow \frac{a}{2} = \frac{b}{-1} = \frac{c}{2}.$$

$\therefore$ d.r.'s of the required line are 2, –1, 2.

And, $\sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3.$

$\therefore$ d.c.'s of the required line are $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$.

EXAMPLE 14 *If* $A(8, 2, 0)$, $B(4, 6, -7)$, $C(-3, 1, 2)$ *and* $D(-9, -2, 4)$ *are four given point then find the angle between* $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$.

SOLUTION D.r.'s of $\overrightarrow{AB}$ are $(4-8), (6-2), (-7-0)$, i.e., –4, 4, –7.

$$|\overrightarrow{AB}| = \sqrt{(-4)^2 + 4^2 + (-7)^2} = \sqrt{16+16+49} = \sqrt{81} = 9.$$

$\therefore$ d.c.'s of $\overrightarrow{AB}$ are $\frac{-4}{9}, \frac{4}{9}, \frac{-7}{9}$.

D.r.'s of $\overrightarrow{CD}$ are $(-9+3), (-2-1), (4-2)$, i.e., –6, –3, 2.

$$|\overrightarrow{CD}| = \sqrt{(-6)^2 + (-3)^2 + 2^2} = \sqrt{36+9+4} = \sqrt{49} = 7.$$

$\therefore$ d.c.'s of $\overrightarrow{CD}$ are $\frac{-6}{7}, \frac{-3}{7}, \frac{2}{7}$.

Let θ be the acute angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Then,

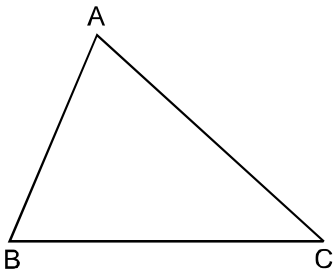
$$\cos\theta = \left|\left(\frac{-4}{9}\right)\times\left(\frac{-6}{7}\right) + \left(\frac{4}{9}\right)\times\left(\frac{-3}{7}\right) + \left(\frac{-7}{9}\right)\times\left(\frac{2}{7}\right)\right|$$

$$= \left| \frac{24}{63} - \frac{12}{63} - \frac{14}{63} \right| = \left| \frac{-2}{63} \right| = \frac{2}{63}.$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{2}{63}\right).$$

Hence, the required angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is $\cos^{-1}\left(\frac{2}{63}\right)$.

EXAMPLE 15 *Find the angles of* $\triangle ABC$ *whose vertices are* $A(-1, 3, 2)$, $B(2, 3, 5)$ *and* $C(3, 5, -2)$.



SOLUTION D.r.'s of AB are $(2+1), (3-3), (5-2)$, i.e. $3, 0, 3$.

D.r.'s of AC are $(3+1), (5-3), (-2-2)$, i.e., $4, 2, -4$.

$$\therefore \quad \cos A = \frac{\{3\times 4 + 0\times 2 + 3\times(-4)\}}{\left\{\sqrt{3^2+0^2+3^2}\right\}\cdot\left\{\sqrt{4^2+2^2+(-4)^2}\right\}} = 0.$$

$$\therefore \quad \angle A = \frac{\pi}{2}.$$

D.r.'s of BA are $(-1-2), (3-3), (2-5)$, i.e., $-3, 0, -3$.

D.r.'s of BC are $(3-2), (5-3), (-2-5)$, i.e., $1, 2, -7$.

$$\therefore \quad \cos B = \frac{\{(-3)\times 1 + 0\times 2 + (-3)\times(-7)\}}{\{\sqrt{(-3)^2+0^2+(-3)^2}\}\cdot\{\sqrt{1^2+2^2+(-7)^2}\}}$$

$$= \frac{18}{\sqrt{18}\times\sqrt{54}} = \frac{1}{\sqrt{3}}.$$

$$\therefore \quad \angle B = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

D.r.'s of CB are $(2-3), (3-5), (5+2)$, i.e., $-1, -2, 7$.

D.r.'s of CA are $(-1-3), (3-5), (2+2)$, i.e., $-4, -2, 4$.

$$\therefore \quad \cos C = \frac{\{(-1)\times(-4) + (-2)\times(-2) + 7\times 4\}}{\{\sqrt{(-1)^2+(-2)^2+7^2}\}\cdot\{\sqrt{(-4)^2+(-2)^2+4^2}\}}$$

$$= \frac{36}{\sqrt{54}\times\sqrt{36}} = \frac{\sqrt{36}}{\sqrt{54}} = \sqrt{\frac{36}{54}} = \sqrt{\frac{2}{3}}.$$

$$\therefore \quad C = \cos^{-1}\left(\sqrt{\frac{2}{5}}\right).$$

Hence, $\angle A = \frac{\pi}{2}$, $\angle B = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\angle C = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$.

EXAMPLE 16 *Show that the points* $A(2, 3, -4)$, $B(1, -2, 3)$ *and* $C(3, 8, -11)$ *are collinear.*

SOLUTION D.r.'s of the line AB are $(1-2), (-2-3), (3+4)$, i.e., $-1, -5, 7$.

D.r.'s of the AC are $(3-2), (8-3), (-11+4)$, i.e., $1, 5, -7$, i.e., $-1, -5, 7$.

$\therefore$ d.r.'s of AB and AC are the same and so they are parallel.

But, the lines AB and AC have a common point A.

Hence, the points A, B and C are collinear.

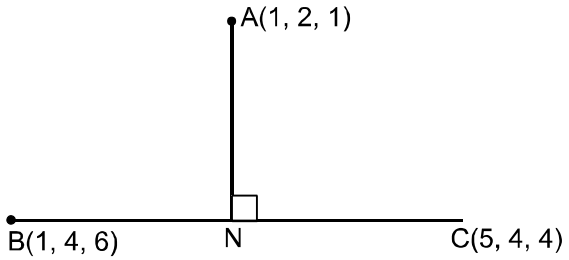
EXAMPLE 17 *Find the coordinates of the foot of the perpendicular drawn from the point* $A(1, 2, 1)$ *to the line joining* $B(1, 4, 6)$ *and* $C(5, 4, 4)$.

SOLUTION Draw $AN \perp BC$.

Let N divide BC in the ratio $k : 1$.

Then, the coordinates of N are

$$\left(\frac{5k+1}{k+1}, \frac{4k+4}{k+1}, \frac{4k+6}{k+1}\right).$$



$\therefore$ d.r.'s of AN are

$$\left(\frac{5k+1}{k+1}-1, \frac{4k+4}{k+1}-2, \frac{4k+6}{k+1}-1\right), \text{ i.e., } \left(\frac{4k}{k+1}, \frac{2k+2}{k+1}, \frac{3k+5}{k+1}\right)$$

And, the d.r.'s of BC are $(5-1), (4-4), (4-6)$, i.e., $4, 0, -2$.

Since $AN \perp BC$, we have

$$\left(4 \times \frac{4k}{k+1}\right) + \left(0 \times \frac{2k+2}{k+1}\right) - \left(2 \times \frac{3k+5}{k+1}\right) = 0$$

$\Rightarrow \quad 16k + 0 - 6k - 10 = 0 \Rightarrow 10k = 10 \Rightarrow k = 1.$

Putting $k = 1$, we get the coordinates of N as $N(3, 4, 5)$.

EXAMPLE 18 *If* l_1, m_1, n_1 *and* l_2, m_2, n_2 *be the direction ratios of two lines then show that the direction ratios of a line which is perpendicular to each of the given lines are:* $(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1)$.

SOLUTION Let a, b, c be the direction ratios of the required line.

Then, this line being perpendicular to each of the given two lines, we have:

$$l_1a + m_1b + n_1c = 0 \quad \text{... (i)}$$

$$l_2a + m_2b + n_2c = 0. \quad \text{... (ii)}$$

On solving (i) and (ii) for a, b, c by cross multiplication, we have

$$\frac{a}{(m_1n_2 - m_2n_1)} = \frac{b}{(n_1l_2 - n_2l_1)} = \frac{c}{(l_1m_2 - l_2m_1)}.$$

Hence, the d.r.'s. of the required line are

$$(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1).$$

EXAMPLE 19 *Find the angle between the two lines whose direction cosines are connected by the relations* $l + m + n = 0$ *and* $l^2 + m^2 - n^2 = 0$.

SOLUTION The given relations are

$$l + m + n = 0 \quad \text{... (i)}$$

$$l^2 + m^2 - n^2 = 0. \quad \text{... (ii)}$$

From (i), we get $l = -(m + n)$.

Putting $l = -(m + n)$ in (ii), we get

$$\{-(m+n)\}^2 + m^2 - n^2 = 0 \Rightarrow 2m^2 + 2mn = 0$$

$$\Rightarrow 2m(m+n) = 0$$

$$\Rightarrow m = 0 \text{ or } (m+n) = 0$$

$$\Rightarrow m = 0 \text{ or } m = -n.$$

Putting $m = 0$ in (i), we get $l = -n$.

Putting $m = -n$ in (i), we get $l = 0$.

$\therefore$ d.r.'s of these lines are $-n, 0, n$ and $0, -n, n$, i.e., $-1, 0, 1$ and $0, -1, 1$.

If θ is the acute angle between these lines, we have

$$\cos\theta = \frac{|(-1)\times 0 + 0\times(-1) + 1\times 1|}{\{\sqrt{(-1)^2 + 0^2 + 1^2}\}\cdot\{\sqrt{0^2 + (-1)^2 + 1^2}\}} = \frac{1}{(\sqrt{2}\times\sqrt{2})} = \frac{1}{2}.$$

$$\therefore \quad \theta = \frac{\pi}{3}.$$

Hence, the angle between the given lines is $\frac{\pi}{3}$.

EXAMPLE 20 *Find the direction cosines of the lines which are connected by the relations* $l - 5m + 3n = 0$ *and* $7l^2 + 5m^2 - 3n^2 = 0$.

SOLUTION The given equations are

$$l - 5m + 3n = 0 \quad \text{... (i)}$$

$$7l^2 + 5m^2 - 3n^2 = 0. \quad \text{... (ii)}$$

Putting $l = (5m - 3n)$ from (i) in (ii), we get

$$7(5m - 3n)^2 + 5m^2 - 3n^2 = 0$$

$$\Rightarrow \quad 6m^2 - 7mn + 2n^2 = 0$$

$$\Rightarrow \quad 6\left(\frac{m}{n}\right)^2 - 7\left(\frac{m}{n}\right) + 2 = 0 \;\Rightarrow\; 6p^2 - 7p + 2 = 0, \text{ where } \frac{m}{n} = p$$

$$\Rightarrow \quad (3p - 2)(2p - 1) = 0$$

$$\Rightarrow \quad p = \frac{2}{3} \text{ or } p = \frac{1}{2} \;\Rightarrow\; \frac{m}{n} = \frac{2}{3} \text{ or } \frac{m}{n} = \frac{1}{2}.$$

$$\text{Now, } \frac{m}{n} = \frac{2}{3} \;\Rightarrow\; \frac{m}{2} = \frac{n}{3} = \frac{5m - 3n}{5 \times 2 - 3 \times 3} = \frac{l}{1}$$

$$\Rightarrow \quad \frac{l}{1} = \frac{m}{2} = \frac{n}{3} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{14}}$$

$$\Rightarrow \quad l = \frac{1}{\sqrt{14}}, m = \frac{2}{\sqrt{14}}, n = \frac{3}{\sqrt{14}}.$$

$$\text{Again, } \frac{m}{n} = \frac{1}{2} \;\Rightarrow\; \frac{m}{1} = \frac{n}{2} = \frac{5m - 3n}{5 \times 1 - 3 \times 2} = \frac{l}{-1}$$

$$\Rightarrow \frac{l}{-1} = \frac{m}{1} = \frac{n}{2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{(-1)^2 + 1^2 + 2^2}} = \frac{1}{\sqrt{6}}$$

$$\Rightarrow l = \frac{-1}{\sqrt{6}}, m = \frac{1}{\sqrt{6}}, n = \frac{2}{\sqrt{6}}.$$

Hence, the direction cosines of the lines are

$$\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) \text{ and } \left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right).$$

EXAMPLE 21 *If a variable line in two adjacent positions has direction cosines (l, m, n) and $(l + \delta l, m + \delta m, n + \delta n)$, show that the small angle $\delta\theta$ between the two positions is given by*

$$(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2.$$

SOLUTION Clearly, we have

$$l^2 + m^2 + n^2 = 1 \qquad \ldots \text{(i)}$$

$$\text{and } (l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1. \qquad \ldots \text{(ii)}$$

Subtracting (i) from (ii), we get

$$(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 - (l^2 + m^2 + n^2) = 0$$

$$\Rightarrow \quad (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = -2(l \cdot \delta l + m \cdot \delta m + n \cdot \delta n) \qquad \ldots \text{(iii)}$$

$$\therefore \quad \cos\delta\theta = l \cdot (l + \delta l) + m \cdot (m + \delta m) + n \cdot (n + \delta n)$$

$$= (l^2 + m^2 + n^2) + (l \cdot \delta l + m \cdot \delta m + n \cdot \delta n)$$

$$= 1 - \frac{1}{2}\{(\delta l)^2 + (\delta m)^2 + (\delta n)^2\} \qquad \text{[using (i) and (iii)].}$$

$$\therefore \quad (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = 2(1 - \cos \delta\theta)$$

$$= 4 \sin^2 \frac{\delta\theta}{2} = 4 \cdot \left(\frac{\delta\theta}{2}\right)^2$$

$$\left[\because \ \frac{\delta\theta}{2} \text{ being small, } \ \sin \frac{\delta\theta}{2} = \frac{\delta\theta}{2}\right]$$

$$= (\delta\theta)^2.$$

Hence, $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

EXAMPLE 22 *Prove that the straight lines whose direction cosines are given by the relations* $al + bm + cn = 0$ *and* $fmn + gnl + hlm = 0$ *are*

perpendicular to each other if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$, *and*

parallel if $a^2f^2 + b^2g^2 + c^2h^2 - 2bcgh - 2cahf - 2abfg = 0$.

SOLUTION The given equations are

$$al + bm + cn = 0 \qquad \ldots \text{(i)}$$

$$fmn + gnl + hlm = 0. \qquad \ldots \text{(ii)}$$

Putting $n = \frac{-(al + bm)}{c}$ from (i) in (ii), we get

$$fm \cdot \left\{\frac{-(al + bm)}{c}\right\} + gl \cdot \left\{\frac{-(al + bm)}{c}\right\} + hlm = 0$$

$$\Rightarrow \quad agl^2 + (af + bg - ch)\, lm + bfm^2 = 0$$

$$\Rightarrow \quad ag\left(\frac{l}{m}\right)^2 + (af + bg - ch) \cdot \left(\frac{l}{m}\right) + bf = 0 \qquad \ldots \text{(iii)}$$

Now, equation (iii), being a quadratic equation in $\left(\frac{l}{m}\right)$, will have two roots, say $\left(\frac{l_1}{m_1}\right)$ and $\left(\frac{l_2}{m_2}\right)$.

$$\therefore \quad \frac{l_1}{m_1} \times \frac{l_2}{m_2} = \frac{bf}{ag} \ \Rightarrow \ \frac{l_1 l_2}{bf} = \frac{m_1 m_2}{ag}$$

$$\Rightarrow \ \frac{l_1 l_2}{\left(\frac{f}{a}\right)} = \frac{m_1 m_2}{\left(\frac{g}{b}\right)} = \frac{n_1 n_2}{\left(\frac{h}{c}\right)} = k \qquad \text{[by symmetry].}$$

$$\therefore \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = k\left(\frac{f}{a} + \frac{g}{b} + \frac{h}{c}\right).$$

Thus, the given lines will be perpendicular to each other

$$\Leftrightarrow \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \ \Leftrightarrow \ \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0.$$

The lines will be parallel only when the roots of (iii) are equal.

$\therefore \quad (af + bg - ch)^2 - 4abgf = 0$

$\Leftrightarrow \quad a^2f^2 + b^2g^2 + c^2h^2 - 2bcgh - 2cahf - 2abfg = 0.$

EXAMPLE 23 *Show that the straight lines whose direction cosines are given by the equations* $al + bm + cn = 0$ *and* $ul^2 + vm^2 + wn^2 = 0$ *are mutually perpendicular if* $a^2(v+w) + b^2(u+w) + c^2(u+v) = 0$, *and parallel if* $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0.$

SOLUTION The given equations are

$$al + bm + cn = 0 \qquad \ldots \text{(i)}$$

$$ul^2 + vm^2 + wn^2 = 0. \qquad \ldots \text{(ii)}$$

Putting $l = \frac{-(bm + cn)}{a}$ from (i) in (ii), we get

$$\frac{u(bm + cn)^2}{a^2} + vm^2 + wn^2 = 0$$

$$\Rightarrow \quad (b^2u + a^2v)m^2 + 2ubcmn + (c^2u + a^2w)n^2 = 0$$

$$\Rightarrow \quad (b^2u + a^2v)\left(\frac{m}{n}\right)^2 + 2ubc\left(\frac{m}{n}\right) + (c^2u + a^2w) = 0. \qquad \ldots \text{(iii)}$$

Let $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ be the roots of (iii).

Then, $\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{c^2u + a^2w}{b^2u + a^2v}$

$$\Rightarrow \quad \frac{m_1m_2}{c^2u + a^2w} = \frac{n_1n_2}{b^2u + a^2v} = \frac{l_1l_2}{b^2w + c^2v} = k \quad \text{[by symmetry].}$$

$$\therefore \quad l_1l_2 + m_1m_2 + n_1n_2 = k(b^2w + c^2v + c^2u + a^2w + b^2u + a^2v).$$

The given lines are mutually perpendicular

$$\Leftrightarrow \quad l_1l_2 + m_1m_2 + n_1n_2 = 0$$

$$\Leftrightarrow \quad a^2(v+w) + b^2(w+u) + c^2(u+v) = 0.$$

For the given lines to be parallel, the direction cosines must be equal.

$\therefore$ the roots of (iii) must be equal.

$$\therefore \quad 4u^2b^2c^2 - 4(b^2u + a^2v)(c^2u + a^2w) = 0 \Leftrightarrow \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0.$$

EXAMPLE 24 *If the edges of a rectangular parallelepiped are a, b, c, prove that the angles between the four diagonals are given by* $\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right)$.

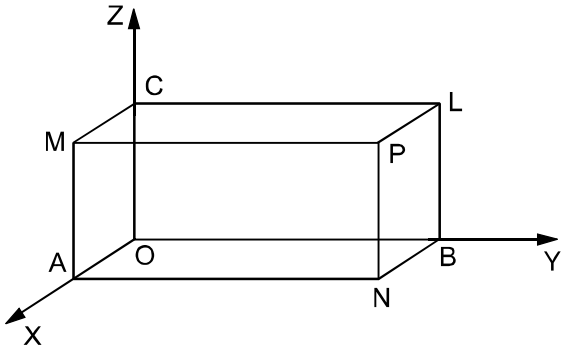
SOLUTION Let OA, OB, OC be the coterminous edges of the parallelepiped, taken along the axes in such a way that $OA = a, OB = b$ and $OC = c$. Then, the coordinates of the vertices are

$O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$,

$C(0, 0, c)$, $P(a, b, c)$, $L(0, b, c)$,

$M(a, 0, c)$ and $N(a, b, 0)$.

The direction ratios of the diagonals OP, AL, BM and CN are $(a, b, c), (-a, b, c), (a, -b, c)$ and $(a, b, -c)$ respectively.



$\therefore$ the direction cosines of OP, AL, BM and CN are

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right),$$

$$\left(\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right),$$

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right),$$

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}\right) \text{ respectively.}$$

Let θ_1 be the angle between OP and AL. Then,

$$\cos\theta_1 = \frac{(-a^2+b^2+c^2)}{(a^2+b^2+c^2)} \quad \text{or} \quad \theta_1 = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right).$$

Again, let θ_2 be the angle between OP and BM. Then,

$$\cos\theta_2 = \frac{(a^2-b^2+c^2)}{(a^2+b^2+c^2)} \quad \text{or} \quad \theta_2 = \cos^{-1}\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right).$$

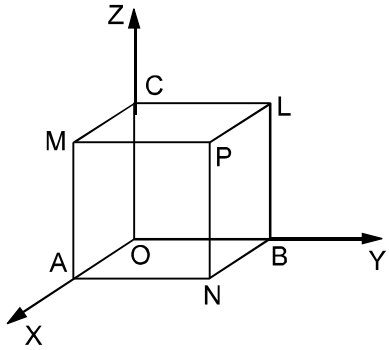
Similarly, the angles between the other pairs of diagonals can be obtained.

Clearly, the angles between the four diagonals can be given by

$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right).$$

EXAMPLE 25 *Show that the angle between any two diagonals of a cube is* $\cos^{-1}\left(\frac{1}{3}\right)$.

SOLUTION Let OA, OB, OC be the coterminous edges of a cube, taken along the axes in such a way that $OA = OB = OC = a$.



Then, the coordinates of the vertices of the cube are

$O(0, 0, 0)$, $A(a, 0, 0,)$, $B(0, a, 0)$,
$C(0, 0, a)$, $P(a, a, a,)$, $L(0, a, a)$,
$M(a, 0, a)$ and $N(a, a, 0)$.

The direction ratios of the diagonals OP, AL, BM and CN are (a, a, a), $(-a, a, a)$, $(a, -a, a)$ and $(a, a, -a)$ respectively.

Thus, direction cosines of OP, AL, BM and CN are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and}$$

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \text{ respectively.}$$

If θ_1 be the angle between OP and AL then

$$\cos\theta_1 = \left\{\frac{1}{\sqrt{3}} \cdot \left(\frac{-1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\right\} = \frac{1}{3}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{1}{3}\right).$$

Similarly, the angle between each one of the other pairs is $\cos^{-1}\left(\frac{1}{3}\right)$.

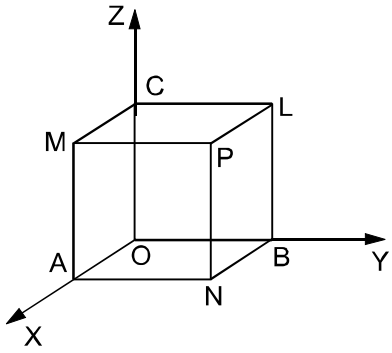
Hence, the angle between any two diagonals of the cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 26 *A line makes angles* $\alpha, \beta, \gamma, \delta$ *with the four diagonals of a cube. Prove that*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}. \qquad \textbf{[CBSE 2007]}$$

SOLUTION Let OA, OB, OC be the coterminous edges of a cube, taken along the axes in such a way that $OA = OB = OC = a$.

Then, the coordinates of the vertices of the cube are $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$, $P(a, a, a)$, $L(0, a, a)$, $M(a, 0, a)$ and $N(a, a, 0)$.



The direction ratios of the diagonals OP, AL, BM and CN are $(a, a, a), (-a, a, a), (a, -a, a)$ and $(a, a, -a)$ respectively.

$\therefore$ the direction cosines of OP, AL, BM and CN are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and } \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \text{ respectively.}$$

Let (l, m, n) be the direction cosines of a line which makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of the cube. Then,

$$\cos\alpha = \left(l\cdot\frac{1}{\sqrt{3}} + m\cdot\frac{1}{\sqrt{3}} + n\cdot\frac{1}{\sqrt{3}}\right) = \frac{(l+m+n)}{\sqrt{3}},$$

$$\cos\beta = \left\{l\cdot\left(\frac{-1}{\sqrt{3}}\right) + m\cdot\frac{1}{\sqrt{3}} + n\cdot\frac{1}{\sqrt{3}}\right\} = \frac{(-l+m+n)}{\sqrt{3}},$$

$$\cos\gamma = \left\{l\cdot\frac{1}{\sqrt{3}} + m\cdot\left(\frac{-1}{\sqrt{3}}\right) + n\cdot\frac{1}{\sqrt{3}}\right\} = \frac{(l-m+n)}{\sqrt{3}},$$

$$\cos\delta = \left\{l\cdot\frac{1}{\sqrt{3}} + m\cdot\frac{1}{\sqrt{3}} + n\cdot\left(\frac{-1}{\sqrt{3}}\right)\right\} = \frac{(l+m-n)}{\sqrt{3}}.$$

On squaring and adding, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\cdot\{(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2\}$$

$$= \frac{4}{3}.$$

EXERCISE 26

1. Find the direction cosines of a line segment whose direction ratios are:
   (i) 2, –6, 3 (ii) 2, –1, –2 (iii) –9, 6, –2
2. Find the direction ratios and the direction cosines of the line segment joining the points:
   (i) $A(1, 0, 0)$ and $B(0, 1, 1)$ **[CBSE 2011]**
   (ii) $A(5, 6, -3)$ and $B(1, -6, 3)$
   (iii) $A(-5, 7, -9)$ and $B(-3, 4, -6)$

3. Show that the line joining the points $A(1, -1, 2)$ and $B(3, 4, -2)$ is perpendicular to the line joining the points $C(0, 3, 2)$ and $D(3, 5, 6)$.

4. Show that the line segment joining the origin to the point $A(2, 1, 1)$ is perpendicular to the line segment joining the points $B(3, 5, -1)$ and $C(4, 3, -1)$.

5. Find the value of p for which the line through the points $A(4, 1, 2)$ and $B(5, p, 0)$ is perpendicular to the line through the points $C(2, 1, 1)$ and $D(3, 3, -1)$.

6. If O be the origin and $P(2, 3, 4)$ and $Q(1, -2, 1)$ be any two points, show that $OP \perp OQ$.

7. Show that the line segment joining the points $A(1, 2, 3)$ and $B(4, 5, 7)$ is parallel to the line segment joining the points $C(-4, 3, -6)$ and $D(2, 9, 2)$.

8. If the line segment joining the points $A(7, p, 2)$ and $B(q, -2, 5)$ be parallel to the line segment joining the points $C(2, -3, 5)$ and $D(-6, -15, 11)$, find the values of p and q.

9. Show that the points $A(2, 3, 4)$, $B(-1, -2, 1)$ and $C(5, 8, 7)$ are collinear.

10. Show that the points $A(-2, 4, 7)$, $B(3, -6, -8)$ and $C(1, -2, -2)$ are collinear.

11. Find the value of p for which the points $A(-1, 3, 2)$, $B(-4, 2, -2)$, and $C(5, 5, p)$ are collinear.

12. Find the angle between the two lines whose direction cosines are:

$$\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \text{ and } \frac{3}{7}, \frac{2}{7}, \frac{6}{7}.$$

13. Find the angle between the two lines whose direction ratios are:

$$a, b, c \text{ and } (b-c), (c-a), (a-b).$$

14. Find the angle between the lines whose direction ratios are:

$$2, -3, 4 \text{ and } 1, 2, 1.$$

15. Find the angle between the lines whose direction ratios are:

$$1, 1, 2 \text{ and } (\sqrt{3}-1), (-\sqrt{3}-1), 4.$$

16. Find the angle between the vectors $\vec{r}_1 = (3\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r}_2 = (4\hat{i} + 5\hat{j} + 7\hat{k})$.

17. Find the angles made by the following vectors with the coordinate axes:

(i) $(\hat{i} - \hat{j} + \hat{k})$ (ii) $(\hat{j} - \hat{k})$ (iii) $(\hat{i} - 4\hat{j} + 8\hat{k})$

18. Find the coordinates of the foot of the perpendicular drawn from the point $A(1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$.

[CBSE 2004, '05]

ANSWERS (EXERCISE 26)

1. (i) $\frac{2}{7}, \frac{-6}{7}, \frac{3}{7}$ (ii) $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$ (iii) $\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$

2. (i) d.r.'s are –1, 1, 1 and d.c.'s are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

(ii) d.r.'s are 2, 6, –3 and d.c.'s are $\frac{2}{7}, \frac{6}{7}, \frac{-3}{7}$.

(iii) d.r.'s are 2, –3, 3 and d.c.'s are $\frac{2}{\sqrt{22}}, \frac{-3}{\sqrt{22}}, \frac{3}{\sqrt{22}}$.

5. $p = \frac{-3}{2}$ **8.** $p = 4, q = 3$ **11.** $p = 10$ **12.** $\cos^{-1}\left(\frac{-8}{21}\right)$

13. $\frac{\pi}{2}$ **14.** $\frac{\pi}{2}$ **15.** $\frac{\pi}{3}$ **16.** $\cos^{-1}\left(\frac{3}{2\sqrt{35}}\right)$

17. (i) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right), \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right), \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (ii) $\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}$

(iii) $\cos^{-1}\left(\frac{1}{9}\right), \cos^{-1}\left(\frac{4}{9}\right), \cos^{-1}\left(\frac{8}{9}\right)$

18. $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$

HINTS TO SOME SELECTED QUESTIONS

1. (ii) D.r.'s of AB are $(1-5), (-6-6), (3+3)$, i.e., –4, –12, 6, i.e., 2, 6, –3.

$\sqrt{2^2 + 6^2 + (-3)^2} = \sqrt{4 + 36 + 9} = \sqrt{49} = 7.$

$\therefore$ d.c.'s of AB are $\frac{2}{7}, \frac{6}{7}, \frac{-3}{7}$.

7. $a_1 = (4-1) = 3, b_1 = (5-2) = 3, c_1 = (7-3) = 4.$

$a_2 = (2+4) = 6, b_2 = (9-3) = 6, c_2 = (2+6) = 8.$

$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$. Hence, $AB \parallel CD$.

8. $a_1 = (q-7), b_1 = (-2-p), c_1 = (5-2) = 3.$

$a_2 = (-6-2) = -8, b_2 = (-15+3) = -12, c_2 = (11-5) = 6.$

Since $AB \parallel CD$, we have $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

$\therefore \quad \frac{q-7}{-8} = \frac{-2-p}{-12} = \frac{3}{6} = \frac{1}{2}$. Find p and q.

9. D.r.'s of AB are $(-1-2), (-2,-3), (1-4)$, i.e., –3, –5, –3, i.e., 3, 5, 3.

D.r.'s. of AC are $(5-2), (8-3), (7-4)$, i.e., 3, 5, 3.

$\therefore \quad AB \parallel AC$ and both have a common end point A.

Hence, A, B and C are collinear.

10. D.r.'s of AB are $(3+2), (-6-4), (-8-7)$, i.e., 5, –10, –15, i.e., 1, –2, –3.
D.r.'s of AC are $(1+2), (-2-4), (-2-7)$, i.e., 3, – 6, – 9, i.e., 1, –2, –3.
$\therefore$ $AB \| AC$ and both have a common end point A.
Hence, A, B and C are collinear.

11. D.r.'s of AB are $(-4+1), (2-3), (-2-2)$, i.e., –3, – 1, – 4, i.e., 3, 1, 4.
D.r.'s of AC are $(5+1), (5-3), (p-2)$, i.e., 6, 2, $p-2$.
Since A, B and C are collinear, we must have $AB \| AC$.
$\therefore$ $\dfrac{3}{6} = \dfrac{1}{2} = \dfrac{4}{p-2} \Rightarrow p-2 = 8 \Rightarrow p = 10.$

12. $\cos\theta = \left[\left(\dfrac{2}{3}\times\dfrac{3}{7}\right) + \left(\dfrac{-1}{3}\times\dfrac{2}{7}\right) + \left(\dfrac{-2}{3}\times\dfrac{6}{7}\right)\right] = \left[\dfrac{6}{21} - \dfrac{2}{21} - \dfrac{12}{21}\right] = \dfrac{(6-14)}{21} = \dfrac{-8}{21}.$
$\therefore$ $\theta = \cos^{-1}\left(\dfrac{-8}{21}\right).$

13. $\cos\theta = \dfrac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{\Sigma a^2}\sqrt{\Sigma(b-c)^2}} = 0 \Rightarrow \theta = \dfrac{\pi}{2}.$

14. $\sqrt{2^2+(-3)^2+4^2} = \sqrt{4+9+16} = \sqrt{29}$ and $\sqrt{1^2+2^2+1^2} = \sqrt{6}.$
$\therefore$ $\cos\theta = \left(\dfrac{2}{\sqrt{29}}\times\dfrac{1}{\sqrt{6}}\right) + \left(\dfrac{-3}{\sqrt{29}}\times\dfrac{2}{\sqrt{6}}\right) + \left(\dfrac{4}{\sqrt{29}}\times\dfrac{1}{\sqrt{6}}\right)$
$= \dfrac{2}{\sqrt{174}} - \dfrac{6}{\sqrt{174}} + \dfrac{4}{\sqrt{174}} = \dfrac{0}{\sqrt{174}} = 0.$
$\therefore$ $\theta = \dfrac{\pi}{2}.$

15. $\sqrt{1^2+1^2+2^2} = \sqrt{6}$ and $\sqrt{(\sqrt{3}-1)^2+(-\sqrt{3}-1)^2+4^2} = \sqrt{24}.$
$\therefore$ $\cos\theta = \left\{\dfrac{1}{\sqrt{6}}\times\dfrac{(\sqrt{3}-1)}{\sqrt{24}} + \dfrac{1}{\sqrt{6}}\times\dfrac{(-\sqrt{3}-1)}{\sqrt{24}} + \dfrac{2}{\sqrt{6}}\times\dfrac{4}{\sqrt{24}}\right\}$
$= \dfrac{(\sqrt{3}-1)}{12} + \dfrac{(-\sqrt{3}-1)}{12} + \dfrac{8}{12} = \dfrac{6}{12} = \dfrac{1}{2}$
$\therefore$ $\theta = \dfrac{\pi}{3}.$

16. $\cos\theta = \dfrac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|} = \dfrac{(3\hat{i}-2\hat{j}+\hat{k})\cdot(4\hat{i}+5\hat{j}+7\hat{k})}{\{\sqrt{3^2+(-2)^2+1^2}\}\{\sqrt{4^2+5^2+7^2}\}} = \dfrac{3}{2\sqrt{35}}.$

17. (i) Let $\vec{a} = \hat{i} - \hat{j} + \hat{k}.$
$\cos\alpha = \dfrac{(\hat{i}-\hat{j}+\hat{k})\cdot\hat{i}}{|\hat{i}-\hat{j}+\hat{k}||\hat{i}|} = \dfrac{1}{\sqrt{3}}$, $\cos\beta = \dfrac{(\hat{i}-\hat{j}+\hat{k})\cdot\hat{j}}{|\hat{i}-\hat{j}+\hat{k}||\hat{j}|} = \dfrac{-1}{\sqrt{3}}$
and $\cos\gamma = \dfrac{(\hat{i}-\hat{j}+\hat{k})\cdot\hat{k}}{|\hat{i}-\hat{j}+\hat{k}||\hat{k}|} = \dfrac{1}{\sqrt{3}}.$

27. STRAIGHT LINE IN SPACE

Equation of a Line Passing through a Given Point and Parallel to a Given Vector

Vector Form

THEOREM 1 *The vector equation of a straight line passing through a given point with position vector $\vec{r_1}$ and parallel to a given vector $\vec{m}$ is $\vec{r} = \vec{r_1} + \lambda \vec{m}$, where λ is a scalar.*

PROOF Let L be the line, passing through a given point A with position vector $\vec{r_1}$ and parallel to a given vector $\vec{m}$.

Let O be the origin. Then, $\overrightarrow{OA} = \vec{r_1}$.

Let P be an arbitrary point on L, and let the position vector of P be $\vec{r}$.

Then, $\overrightarrow{OP} = \vec{r}$.

Clearly, $\overrightarrow{AP} \,||\, \vec{m}$

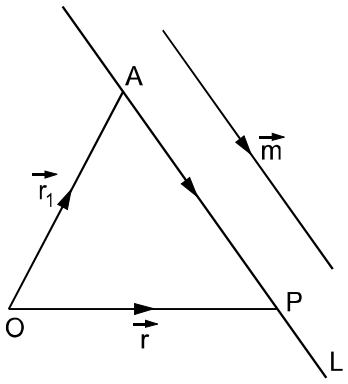
$\Rightarrow \quad \overrightarrow{AP} = \lambda \vec{m}$, for some scalar λ

$\Rightarrow \quad$ (p.v. of P) – (p.v. of A) $= \lambda \vec{m}$

$\Rightarrow \quad \overrightarrow{OP} - \overrightarrow{OA} = \lambda \vec{m}$

$\Rightarrow \quad \vec{r} - \vec{r_1} = \lambda \vec{m}$

$\Rightarrow \quad \vec{r} = \vec{r_1} + \lambda \vec{m}$. … (i)



Clearly, every point on the line L satisfies (i), and for any value of λ, (i) gives the position vector of a point P on the line.

Hence, $\vec{r} = \vec{r_1} + \lambda \vec{m}$ is the desired equation.

COROLLARY *The vector equation of a straight line passing through the origin and parallel to a given vector $\vec{m}$ is $\vec{r} = \lambda \vec{m}$.*

PROOF Taking $\vec{r_1} = \vec{0}$ in (i), we get the desired equation, $\vec{r} = \lambda \vec{m}$.

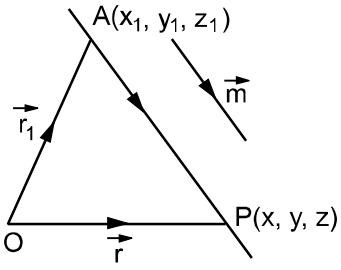
Cartesian Form

THEOREM 2 *The equations of a straight line with direction ratios a, b, c, and passing through a point $A(x_1, y_1, z_1)$ are*

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}.$$

PROOF We know that the vector equation of a straight line passing through a fixed point $A(x_1, y_1, z_1)$ with position vector $\vec{r_1}$ and parallel to a given vector $\vec{m}$ is given by

$\vec{r}=\vec{r_1}+\lambda\vec{m}$ … (i), where λ is a scalar.



Let $P(x, y, z)$ be the given point on line with position vector $\vec{r}$.

Then, $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $\vec{r}_1=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$.

Since the direction ratios of the given line are a, b, c, and this line is parallel to $\vec{m}$, the direction ratios of $\vec{m}$ are a, b, c.

$\therefore\ \vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$.

Putting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $\vec{r_1}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$ in (i), we get the equation of the line as

$$(x\hat{i}+y\hat{j}+z\hat{k})=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k})$$

$$\Rightarrow\ (x\hat{i}+y\hat{j}+z\hat{k})=(x_1+\lambda a)\hat{i}+(y_1+\lambda b)\hat{j}+(z_1+\lambda c)\hat{k}$$

$$\Rightarrow\ x=x_1+\lambda a,\ y=y_1+\lambda b \text{ and } z=z_1+\lambda c$$

$$\Rightarrow\ \frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda.$$

Hence, the equations of a line having direction ratios a, b, c and passing through $A(x_1, y_1, z_1)$ are

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}.$$

COROLLARY *The equations of a line having direction cosines l, m, n and passing through (x_1, y_1, z_1) are*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

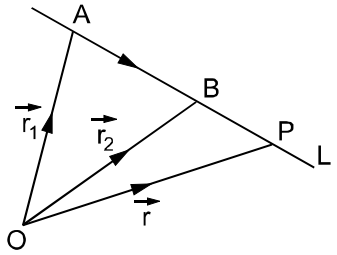
PROOF Since the direction cosines of a line are proportional to the direction ratios of the line, the result follows.

Equation of a Line Passing through Two Given Points

Vector Form

THEOREM 3 *The vector equation of a straight line passing through two points with position vectors $\vec{r_1}$ and $\vec{r_2}$ is given by $\vec{r}=\vec{r_1}+\lambda(\vec{r_2}-\vec{r_1})$.*

PROOF Let L be the given line, passing through two given points A and B with position vectors $\vec{r_1}$ and $\vec{r_2}$ respectively.



Let O be the origin.

Then, $\overrightarrow{OA} = \vec{r_1}$ and $\overrightarrow{OB} = \vec{r_2}$.

$\therefore \quad \overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A) = (\vec{r_2} - \vec{r_1})$.

Let P be an arbitrary point on L, having the position vector $\vec{r}$.

Then, $\overrightarrow{OP} = \vec{r}$.

$\therefore \overrightarrow{AP} = (\text{p.v. of } P) - (\text{p.v. of } A)$

$= (\overrightarrow{OP} - \overrightarrow{OA}) = (\vec{r} - \vec{r_1})$.

Since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear vectors, we have

$\overrightarrow{AP} = \lambda(\overrightarrow{AB})$, for some scalar λ

$\Rightarrow \quad (\vec{r} - \vec{r_1}) = \lambda(\vec{r_2} - \vec{r_1})$

$\Rightarrow \quad \vec{r} = \vec{r_1} + \lambda(\vec{r_2} - \vec{r_1})$.

Hence, the vector equation of a line L, passing through two given points A and B with position vectors $\vec{r_1}$ and $\vec{r_2}$, is given by

$$\vec{r} = \vec{r_1} + \lambda(\vec{r_2} - \vec{r_1}).$$

Cartesian Form

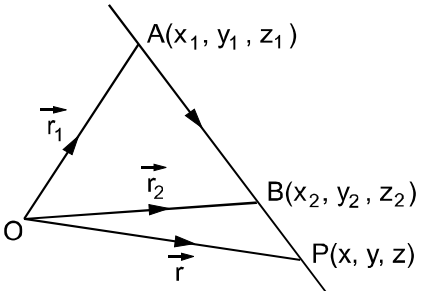
THEOREM 4 *The equations of a line passing through two given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are given by*

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

PROOF We know that the vector equation of a line passing through two points A and B with position vectors $\vec{r_1}$ and $\vec{r_2}$ is given by

$$\vec{r} = \vec{r_1} + \lambda(\vec{r_2} - \vec{r_1}). \qquad \text{... (i)}$$

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the points on the given line with position vectors $\vec{r_1}$ and $\vec{r_2}$ respectively.



Let $P(x, y, z)$ be an arbitrary point on this line with position vector $\vec{r}$.

Then, $\vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Substituting these values in (i), we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

$$\Rightarrow \quad (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1\hat{j} + (z_2 - z_1)\hat{k}]$$

$$\Rightarrow \quad (x - x_1) = \lambda(x_2 - x_1), (y - y_1) = \lambda(y_2 - y_1) \text{ and } (z - z_1) = \lambda(z_2 - z_1)$$

$$\Rightarrow \quad \frac{(x - x_1)}{(x_2 - x_1)} = \frac{(y - y_1)}{(y_2 - y_1)} = \frac{(z - z_1)}{(z_2 - z_1)} (= \lambda),$$

which are the required equations.

SUMMARY

1. (i) If a line passes through a point with p.v. $\vec{r_1}$ and it is parallel to $\vec{m}$ then its vector equation is:

$$\vec{r} = \vec{r_1} + \lambda\vec{m}.$$

(ii) If a line passes through a point $A(x_1, y_1.z_1)$ and it has d.r.'s a, b, c then its Cartesian equations are:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

2. (i) If a line passes through two points having p.v.'s $\vec{r_1}$ and $\vec{r_2}$, then its vector equation is:

$$\vec{r} = \vec{r_1} + \lambda(\vec{r_2} - \vec{r_1}).$$

(ii) If a line passes through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ then its Cartesian equations are:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the vector equation of a line which is parallel to the vector* $(2\hat{i} - \hat{j} + 3\hat{k})$ *and which passes through the point* $(5, -2, 4)$. *Also find its Cartesian equations.* **[CBSE 2007]**

SOLUTION *Vector equation of the given line:*

The given line passes through the point $A(5, -2, 4)$ and it is parallel to the vector $\vec{m} = (2\hat{i} - \hat{j} + 3\hat{k})$.

The position vector of A is given by $\vec{r_1} = (5\hat{i} - 2\hat{j} + 4\hat{k})$.

Hence, the vector equation of the given line is

$$\vec{r} = \vec{r_1} + \lambda\vec{m} \Rightarrow \vec{r} = (5\hat{i} - 2\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k}). \quad \text{... (i)}$$

Cartesisan equation of the given line:

Taking $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$, equation (i) becomes:

$$(x\hat{i} + y\hat{j} + z\hat{k}) = (5\hat{i} - 2\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) = (5 + 2\lambda)\hat{i} - (2 + \lambda)\hat{j} + (4 + 3\lambda)\hat{k}$$

$$\Rightarrow x = 5 + 2\lambda,\ y = -(2 + \lambda) \text{ and } z = 4 + 3\lambda$$

$$\Rightarrow \frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3} = \lambda.$$

Hence, $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$ are the required equations of the given line in Cartesian form.

EXAMPLE 2 *The Cartesian equations of a line are* $3x - 3 = 2y + 1 = 5 - 6z$.

(a) Write these equations in standard form and find the direction ratios of the given line.

(b) Write the equation of the given line in vector form.

SOLUTION (a) The given equations may be written as

$$3(x-1) = 2\left(y + \frac{1}{2}\right) = -6\left(z - \frac{5}{6}\right)$$

$$\Rightarrow \frac{(x-1)}{\left(\frac{1}{3}\right)} = \frac{\left(y+\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = \frac{\left(z-\frac{5}{6}\right)}{\left(\frac{-1}{6}\right)}$$

$$\Rightarrow \frac{(x-1)}{2} = \frac{\left(y+\frac{1}{2}\right)}{3} = \frac{\left(z+\frac{5}{6}\right)}{-1} \quad \text{[on dividing each by 6].}$$

This is the standard form of the given equations in Cartesian form. Clearly, its direction ratios are 2, 3, –1.

(b) The given line passes through the point $A\left(1, \frac{-1}{2}, \frac{5}{6}\right)$ and it is parallel to the vector $\vec{m} = (2\hat{i} + 3\hat{j} - \hat{k})$.

The position vector of the point A is $\vec{r_1} = \left(\hat{i} - \frac{1}{2}\hat{j} + \frac{5}{6}\hat{k}\right)$.

So, the vector equation of the given line is

$$\vec{r} = \vec{r_1} + \lambda \vec{m} \Rightarrow \vec{r} = \left(\hat{i} - \frac{1}{2}\hat{j} + \frac{5}{6}\hat{k}\right) + \lambda(2\hat{i} + 3\hat{j} - \hat{k}).$$

EXAMPLE 3 *The Cartesian equations of a line are* $6x - 2 = 3y + 1 = 2z - 2$.

(a) Write these equations in standard form and find the direction cosines of the given line.

(b) Write down the Cartesian and vector equations of a line, passing through $(2, -1, 1)$ *and parallel to the given line.* **[CBSE 2013C]**

SOLUTION (a) The given equations may be written as

$$6\left(x - \frac{1}{3}\right) = 3\left(y + \frac{1}{3}\right) = 2(z - 1)$$

$$\Rightarrow \quad \frac{\left(x - \frac{1}{3}\right)}{\left(\frac{1}{6}\right)} = \frac{\left(y + \frac{1}{3}\right)}{\left(\frac{1}{3}\right)} = \frac{(z-1)}{\left(\frac{1}{2}\right)}$$

$$\Rightarrow \quad \frac{\left(x - \frac{1}{3}\right)}{1} = \frac{\left(y + \frac{1}{3}\right)}{2} = \frac{(z-1)}{3} \qquad \text{[on dividing each by 6]}$$

This is the standard form of the given equations in Cartesian form. The direction ratios of the given are 1, 2, 3.

Also, $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

So, the direction cosines of the given line are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

(b) The required line passes through the point $A(2, -1, 1)$ and it is parallel to the line having direction ratios 1,2, 3.

The equations of this line in Cartesian form are given by

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-1}{3}. \qquad \text{... (i)}$$

The position vector of the point A is $\vec{r_1} = (2\hat{i} - \hat{j} + \hat{k})$.

The required line is parallel to the vector $\vec{m} = (\hat{i} + 2\hat{j} + 3\hat{k})$.

So, its equation in vector form is

$$\vec{r} = \vec{r_1} + \lambda \vec{m} \Rightarrow \vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k}). \quad \text{... (ii)}$$

Thus, the Cartesian and vector forms of equations of the desired line are given by (i) and (ii) respectively.

EXAMPLE 4 *Find the vector and Cartesian equations of the line passing through the point* $(1, 2, -4)$ *and perpendicular to each of the lines*

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y+29}{8}=\frac{z-5}{-5}.$$ **[CBSE 2012]**

SOLUTION The given lines are

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \quad \text{... (i)}$$

$$\text{and } \frac{x-15}{3}=\frac{y+29}{8}=\frac{z-5}{-5}. \quad \text{... (ii)}$$

Let a, b, c be the direction ratios of the required line.

Then, it being perpendicular to each of the lines (i) and (ii), we have $3a-16b+7c=0$ and $3a+8b-5c=0$.

On solving these equations by cross multiplication, we get

$$\frac{a}{(80-56)}=\frac{b}{(21+15)}=\frac{c}{(24+48)}$$

$$\Rightarrow \quad \frac{a}{24}=\frac{b}{36}=\frac{c}{72} \quad \Rightarrow \quad \frac{a}{2}=\frac{b}{3}=\frac{c}{6}.$$

Thus, the desired line has direction ratios 2, 3, 6.

So, we have to find the equations of a line passing through the point $A(1, 2, -4)$ and having 2, 3, 6 as its direction ratios.

So, the required equations in Cartesian form are

$$\frac{(x-1)}{2}=\frac{(y-2)}{3}=\frac{(z+4)}{6}.$$

The position vector of point A is $\vec{r_1}=(\hat{i}+2\hat{j}-4\hat{k})$.

Also, the required line has direction ratios 2, 3, 6 and so it is parallel to the vector $\vec{m}=(2\hat{i}+3\hat{j}+6\hat{k})$.

So, its equation in vector form is

$$\vec{r}=\vec{r_1}+\lambda\vec{m} \Rightarrow \vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k}).$$

EXAMPLE 5 *Find the equation of a line passing through the point* $P(2,-1,3)$ *and perpendicular to the lines*

$$\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$$

and $\vec{r}=(2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$ **[CBSE 2012]**

SOLUTION The given lines are

$\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ ... (i), where $\vec{a_1}=(\hat{i}+\hat{j}-\hat{k})$ and $\vec{b_1}=(2\hat{i}-2\hat{j}+\hat{k})$ and

$\vec{r}=\vec{a_2}+\lambda\vec{b_2}$ ... (ii), where $\vec{a_2}=(2\hat{i}-\hat{j}-3\hat{k})$ and $\vec{b_2}=(\hat{i}+2\hat{j}+2\hat{k})$.

The required line is perpendicular to (i) as well as (ii).

Also (i) is parallel to $\vec{b_1}$ and (ii) is parallel to $\vec{b_2}$.

So, the required line is perpendicular to both $\vec{b_1}$ and $\vec{b_2}$.

Consequently, this line must be parallel to $(\vec{b_1} \times \vec{b_2})$.

$$\text{Now, } (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = (-6\hat{i} - 3\hat{j} + 6\hat{k}).$$

So, we have to find the equation of a line passing through the point $P(2, -1, 3)$ and parallel to $(\vec{b_1} \times \vec{b_2})$.

Hence, the required equation is

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + t(-6\hat{i} - 3\hat{j} + 6\hat{k}),$$

where t is an arbitrary constant.

EXAMPLE 6 *Find the vector equation of the line passing through the point $A(2, -1, 1)$, and parallel to the line joining the points $B(-1, 4, 1)$ and $C(1, 2, 2)$. Also, find the Cartesian equations of the line.* **[CBSE 2003]**

SOLUTION *Vector equation of the given line*:

The p.v. of $B = (-\hat{i} + 4\hat{j} + \hat{k})$ and p.v. of $C = (\hat{i} + 2\hat{j} + 2\hat{k})$.

$$\therefore \quad \overrightarrow{BC} = (\text{p.v. of } C) - (\text{p.v. of } B)$$

$$= (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 4\hat{j} + \hat{k}) = (2\hat{i} - 2\hat{j} + \hat{k}).$$

The p.v. of A is $\vec{r_1} = 2\hat{i} - \hat{j} + \hat{k}$.

$\therefore$ the vector equation of the given line is

$$\vec{r} = \vec{r_1} + \lambda(\overrightarrow{BC})$$

$$\Leftrightarrow \quad \vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k}) \qquad \ldots \text{(i)}$$

Cartesian equations of the given line:

Taking $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, equation (i) becomes

$$(x\hat{i} + y\hat{j} + z\hat{k}) = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$$

$$\Leftrightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) = (2 + 2\lambda)\hat{i} + (-1 - 2\lambda)\hat{j} + (1 + \lambda)\hat{k}$$

$$\Leftrightarrow \quad x = 2 + 2\lambda, \; y = -1 - 2\lambda \text{ and } z = 1 + \lambda$$

$$\Leftrightarrow \quad \frac{x-2}{2}=\frac{y+1}{-2}=\frac{z-1}{1}=\lambda.$$

Hence, $\frac{x-2}{2}=\frac{y+1}{-2}=\frac{z-1}{1}$ are the required equations of the given line in the Cartesian form.

EXAMPLE 7 *Find the vector and Cartesian equations of the line passing through the points $A(2, -1, 4)$ and $B(1, 1, -2)$.*

SOLUTION *Vector equation of the given line*:

Let the position vectors of A and B be $\vec{r_1}$ and $\vec{r_2}$ respectively. Then,

$$\vec{r_1}=2\hat{i}-\hat{j}+4\hat{k} \text{ and } \vec{r_2}=\hat{i}+\hat{j}-2\hat{k}.$$

$$\therefore \quad (\vec{r_2}-\vec{r_1})=(\hat{i}+\hat{j}-2\hat{k})-(2\hat{i}-\hat{j}+4\hat{k})=(-\hat{i}+2\hat{j}-6\hat{k}).$$

$\therefore$ the vector equation of the line AB is

$$\vec{r}=\vec{r_1}+\lambda(\vec{r_2}-\vec{r_1}) \text{ for some scalar, } \lambda,$$

i.e., $\vec{r}=(2\vec{i}-\vec{j}+4\vec{k})+\lambda(-\vec{i}+2\vec{j}-6\vec{k})$. … (i)

Cartesian equations of the given line:

Taking $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, equation (i) becomes

$$(x\hat{i}+y\hat{j}+z\hat{k})=(2\hat{i}-\hat{j}+4\hat{k})+\lambda(-\hat{i}+2\hat{j}-6\hat{k})$$

$$\Leftrightarrow \quad (x\hat{i}+y\hat{j}+z\hat{k})=(2-\lambda)\hat{i}+(2\lambda-1)\hat{j}+(4-6\lambda)\hat{k}$$

$$\Leftrightarrow \quad x=2-\lambda,\ y=2\lambda-1 \text{ and } z=4-6\lambda$$

$$\Leftrightarrow \quad \frac{x-2}{-1}=\frac{y+1}{2}=\frac{z-4}{-6}=\lambda.$$

Hence, $\frac{x-2}{-1}=\frac{y+1}{2}=\frac{z-4}{-6}$ are the Cartesian equations of the given line.

EXAMPLE 8 *Show that the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect. Also find their point of intersection.* **[CBSE 2014]**

SOLUTION The given lines are

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}=\lambda \text{ (say)} \qquad \ldots (1)$$

and $$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}=\mu \text{ (say)}. \qquad \ldots (2)$$

The general point on (1) is $P(3\lambda-1, 5\lambda-3, 7\lambda-5)$.

The general point on (2) is $Q(\mu + 2,\ 3\mu + 4, 5\mu + 6)$.

The given lines will intersect only when they have a common point. This happens when P and Q coincide for some particular values of λ and μ.

So, the given lines will intersect only when

$$3\lambda - 1 = \mu + 2, 5\lambda - 3 = 3\mu + 4 \text{ and } 7\lambda - 5 = 5\mu + 6$$

$$\Rightarrow \quad 3\lambda - \mu = 3 \ \text{... (i)}, \ 5\lambda - 3\mu = 7 \ \text{... (ii) and } \ 7\lambda - 5\mu = 11. \quad \text{... (iii)}$$

On solving (i) and (ii), we get $\lambda = \frac{1}{2}$ and $\mu = \frac{-3}{2}$.

Clearly, these values of λ and μ also satisfy (iii).

Hence, the given lines intersect.

Putting $\lambda = \frac{1}{2}$ in P or $\mu = \frac{-3}{2}$ in Q, we get the point of intersection of the given lines as $\left(\frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}\right)$.

EXAMPLE 9 *Show that the lines*

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5} \text{ and } \frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$$

do not intersect. **[CBSE 2002]**

SOLUTION The given lines are

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5} = \lambda \text{ (say)} \quad \ldots (1)$$

$$\frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2} = \mu \text{ (say)}. \quad \ldots (2)$$

The general point on (1) is $P(3\lambda + 1, 2\lambda - 1, 5\lambda + 1)$.

The general point on (2) is $Q(4\mu - 2,\ 3\mu + 1, -2\mu - 1)$.

If possible, let the given lines intersect.

Then, P and Q coincide for some particular values of λ and μ.

In that case, we have

$$3\lambda + 1 = 4\mu - 2,\ 2\lambda - 1 = 3\mu + 1 \text{ and } 5\lambda + 1 = -2\mu - 1$$

$$\Leftrightarrow \quad 3\lambda - 4\mu = -3 \ \ldots \text{(i)}, 2\lambda - 3\mu = 2 \ \ldots \text{(ii)}, 5\lambda + 2\mu = -2 \ \ldots \text{(iii)}.$$

Solving (i) and (ii), we get $\lambda = -17$ and $\mu = -12$.

However, these values of λ and μ do not satisfy (iii).

Hence, the given lines do not intersect.

EXAMPLE 10 *Show that the lines*

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) \text{ and } \vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

intersect each other. Find their point of intersection. **[CBSE 2013, '14]**

SOLUTION The given lines are

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) \quad \ldots (1)$$

$$\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k}). \quad \ldots (2)$$

These lines will intersect if for some particular values of λ and μ, the values of $\vec{r}$ given by (1) and (2) are the same.

So, we must have

$$(\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

$$\Rightarrow \quad (1 + 3\lambda)\hat{i} + (1 - \lambda)\hat{j} - \hat{k} = (4 + 2\mu)\hat{i} + (3\mu - 1)\hat{k}$$

$$\Rightarrow \quad 1 + 3\lambda = 4 + 2\mu, 1 - \lambda = 0 \text{ and } 3\mu - 1 = -1$$

$$\Rightarrow \quad 3\lambda - 2\mu = 3 \ \ldots \text{(i)}, \ \lambda = 1 \ \ldots \text{(ii)} \ \text{and} \ \mu = 0 \ \ldots \text{(iii)}.$$

Clearly, $\lambda = 1$ and $\mu = 0$ also satisfy (i).

Hence, the given lines intersect.

Putting $\lambda = 1$ in (1), we get $\vec{r} = (4\hat{i} + 0\hat{j} - \hat{k})$.

Hence, the point of intersection of the given lines is (4, 0, –1).

EXAMPLE 11 *Show that the lines* $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ *and* $\vec{r} = (\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + 2\hat{k})$ *do not intersect.*

SOLUTION The given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \quad \ldots (1)$$

$$\vec{r} = (\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + 2\hat{k}) \quad \ldots (2)$$

These lines will intersect if for some particular values of λ and μ, the values of $\vec{r}$ given by (1) and (2) are the same.

This happens, when

$$(\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) = (\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + 2\hat{k})$$

$$\Rightarrow \quad (1 + \lambda)\hat{i} + (2 - \lambda)\hat{j} + (1 + \lambda)\hat{k} = (1 + \mu)\hat{i} + (1 - \mu)\hat{j} + (1 + 2\mu)\hat{k}$$

$$\Rightarrow \quad 1 + \lambda = 1 + \mu, \ 2 - \lambda = 1 - \mu \text{ and } 1 + \lambda = 1 + 2\mu$$

$$\Rightarrow \quad \lambda - \mu = 0 \quad \ldots \text{(i)}, \quad \lambda - \mu = 1 \quad \ldots \text{(ii)}, \quad \lambda - 2\mu = 0 \quad \ldots \text{(iii)}.$$

From (ii) and (iii), we get $\lambda = 2$ and $\mu = 1$.

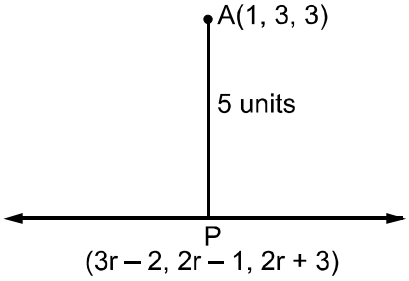
And, these values of λ and μ do not satisfy (i).

Hence, the given lines do not intersect.

EXAMPLE 12 *Find the points on the line* $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ *at a distance of* 5 *units from the point* (1, 3, 3). **[CBSE 2010]**

SOLUTION The given line is

$$\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}=r \text{ (say).} \quad \text{... (i)}$$



The general point on this line is

$$P(3r-2,\ 2r-1,\ 2r+3).$$

The given point is $A(1, 3, 3)$.

Now, $PA=5 \Rightarrow (PA)^2=25$

$$\Rightarrow (3r-2-1)^2+(2r-1-3)^2+(2r+3-3)^2=25$$
$$\Rightarrow (3r-3)^2+(2r-4)^2+(2r)^2=25$$
$$\Rightarrow 17r^2-34r=0 \Rightarrow 17r(r-2)=0$$
$$\Rightarrow r=0 \text{ or } r=2.$$

$r=0 \Rightarrow$ the required point is $P(–2, –1, 3)$.

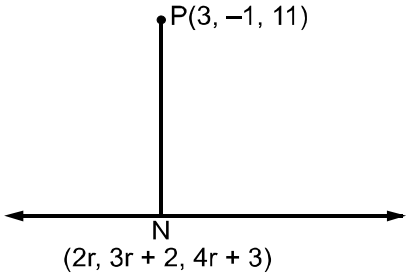
$r=2 \Rightarrow$ the required point is $P(4, 3, 7)$.

Hence, the rquired points are $(–2, –1, 3)$ and $(4, 3, 7)$.

EXAMPLE 13 *Find the equations of the perpendicular from the point* $(3, -1, 11)$ *to the line* $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$. *Also find the coordinates of the foot of the perpendicular and the length of the perpendicular.* **[CBSE 2011C]**

SOLUTION The given line is

$$\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r \text{ (say).} \quad \text{... (i)}$$



The general point on this line is

$$(2r,\ 3r+2,\ 4r+3).$$

Let N be the foot of the perpendicular drawn from the point $P(3, -1, 11)$ on the given line.

Then, this point is $N(2r,\ 3r+2,\ 4r+3)$ for some fixed value of r.

D.r.'s of PN are $(2r-3), (3r+3), (4r-8)$. ... (ii)

D.r.'s of the given line (i) are 2, 3, 4.

Since PN is perpendicular to the given line (i), we have

$$2(2r-3)+3(3r+3)+4(4r-8)=0 \Rightarrow 29r=29 \Rightarrow r=1.$$

$\therefore$ D.r.'s of PN are –1, 6, –4 [putting $r=1$ in (ii)].

So, the required equations of perpendicular PN are

$$\frac{x-3}{-1}=\frac{y+1}{6}=\frac{z-11}{-4}.$$

Now, the coordinates of N are $N(2, 5, 7)$ [*Putting* $r=1$]

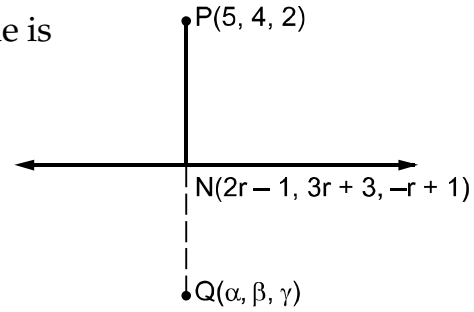
Length of perpendicular PN

$$=\sqrt{(2-3)^2+(5+1)^2+(7-11)^2}=\sqrt{(-1)^2+6^2+(-4)^2}$$
$$=\sqrt{1+36+16}=\sqrt{53}.$$

EXAMPLE 14 *Find the coordinates of the foot of perpendicular and the length of the perpendicular drawn from the point $P(5, 4, 2)$ to the line $\vec{r} = (-\hat{i} + 3\hat{j} + \hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$. Also find the image of P in this line.* **[CBSE 2012]**

SOLUTION The vector equation of the given line is

$$\vec{r} = (-\hat{i} + 3\hat{j} + \hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k}).$$



Clearly, it passes through the point $(-1, 3, 1)$ and it has direction ratios $2, 3, -1$.

So, its Cartesian equations are

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} = r \text{ (say).} \quad \text{... (i)}$$

The general point on this line is $(2r-1,\ 3r+3,\ -r+1)$.

Let N be the foot of the perpendicular drawn from the point $P(5,4,2)$ on the given line.

Then, this point is $N(2r-1,\ 3r+3,\ -r+1)$ for some fixed value of r.

D.r.'s of PN are $(2r-6,\ 3r-1,\ -r-1)$.

D.r.'s of the given line are $2, 3, -1$.

Since PN is perpendicular to the given line (i), we have

$$2(2r-6) + 3(3r-1) - 1\cdot(-r-1) = 0 \Rightarrow 14r = 14 \Rightarrow r = 1.$$

So, the point N is given by $N(1, 6, 0)$.

Hence, the foot of the perpendicular from the given point $P(5,4,2)$ on the given line is $N(1, 6, 0)$.

Let $Q(\alpha, \beta, \gamma)$ be the image of $P(5, 4, 2)$ in the given line.

Then, $N(1, 6, 0)$ is the midpoint of PQ.

$$\therefore \quad \frac{5+\alpha}{2} = 1,\ \frac{4+\beta}{2} = 6 \text{ and } \frac{2+\gamma}{2} = 0 \Rightarrow \alpha = -3,\ \beta = 8 \text{ and } \gamma = -2.$$

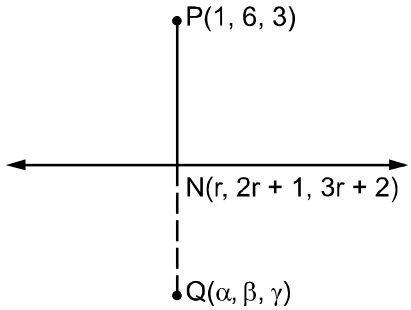
Hence, the image of $P(5, 4, 2)$ in the given line is $Q(-3, 8, -2)$.

EXAMPLE 15 *Find the image of the point $P(1, 6, 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.*

[CBSE 2003C, '08C, '10C]

SOLUTION The equations of the given line are

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = r \text{ (say).} \quad \text{... (i)}$$



The general point on this line is

$$(r,\ 2r+1,\ 3r+2).$$

Let N be the foot of the perpendicular drawn from the point $P(1,6,3)$ on the given line.

Then, this point is $N(r,\ 2r+1,\ 3r+2)$ for some fixed value of r.

D.r.'s of PN are $(r-1,\ 2r-5,\ 3r-1)$.

D.r.'s of the given line are 1, 2, 3.

Since, PN is perpendicular to the given line (i), we have

$$1\cdot(r-1)+2(2r-5)+3(3r-1)=0\Rightarrow 14r=14\Rightarrow r=1.$$

So, the point N is given by $N(1, 3, 5)$.

Let $Q(\alpha, \beta, \gamma)$ be the image of $P(1, 6, 3)$ in the given line.

Then, N is the midpoint of PQ.

$$\therefore\quad \frac{\alpha+1}{2}=1,\ \frac{\beta+6}{2}=3,\ \frac{\gamma+3}{2}=5\Rightarrow \alpha=1,\ \beta=0 \text{ and } \gamma=7.$$

Hence, the image of the point $P(1, 6, 3)$ in the given line is $Q(1, 0, 7)$.

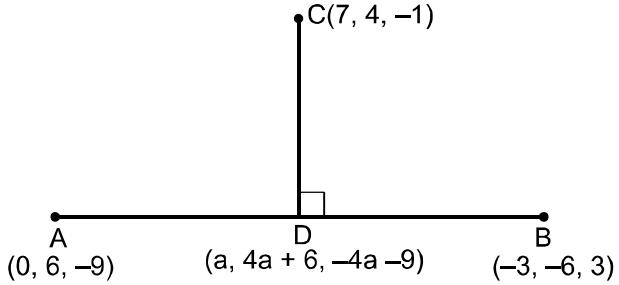
EXAMPLE 16 *Find the equations of the line passing through the points $A(0, 6, -9)$ and $B(-3, -6, 3)$. If D is the foot of the perpendicular drawn from a point $C(7, 4, -1)$ on the line AB then find the coordinates of D and the equations of the line CD.* **[CBSE 2010C]**

SOLUTION The equations of line AB are

$$\frac{x-0}{-3-0}=\frac{y-6}{-6-6}=\frac{z+9}{3+9}\left[\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}\right]$$

$$\Rightarrow\quad \frac{x}{-3}=\frac{y-6}{-12}=\frac{z+9}{12}$$

$$\Rightarrow\quad \frac{x}{1}=\frac{y-6}{4}=\frac{z+9}{-4}=a \text{ (say).} \qquad \text{... (i)}$$



Thus, the required equations of line AB are given by (i).

From (i), we get $x=a$, $y=4a+6$ and $z=-4a-9$.

So, the coordinates of D are $D(a, 4a+6, -4a-9)$ for some particular value of a.

D.r.'s of AB are 1, 4, –4.

D.r.'s of CD are $(a-7), (4a+2), (-4a-8)$.

Since $AB\perp CD$, we have

$$1\cdot(a-7)+4(4a+2)-4(-4a-8)=0\ \Rightarrow\ 33a=-33\ \Rightarrow\ a=-1.$$

Putting $a=-1$, we get the coordinates of D as $D(-1, 2, -5)$.

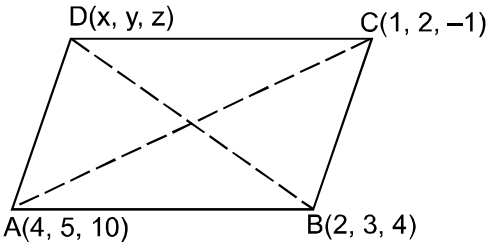
The equations of the line CD are

$$\frac{x-7}{-1-7}=\frac{y-4}{2-4}=\frac{z+1}{-5+1} \Rightarrow \frac{x-7}{-8}=\frac{y-4}{-2}=\frac{z+1}{-4}$$
$$\Rightarrow \frac{x-7}{4}=\frac{y-4}{1}=\frac{z+1}{2}.$$

EXAMPLE 17 *The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of a parallelogram $ABCD$. Find the vector equations of the sides AB and BC and also find the coordinates of point D.* **[CBSE 2010]**

SOLUTION D.r.'s of side AB are $(2-4), (3-5), (4-10)$, i.e., $-2, -2, -6$.

Thus, side AB passes through the point $A(4, 5, 10)$ and it has D.r.'s $-2, -2, -6$.



So, the vector equation of side AB is

$$\vec{r}=(4\hat{i}+5\hat{j}+10\hat{k})+\lambda(-2\hat{i}-2\hat{j}-6\hat{k}).$$

D.r.'s. of side BC are $(1-2), (2-3), (-1-4)$, i.e., $-1, -1, -5$.

Thus, side BC passes through the point $B(2, 3, 4)$ and it has d.r.'s $-1, -1, -5$.

So, the vector equation of side BC is

$$\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+\mu(-\hat{i}-3\hat{j}-5\hat{k}).$$

Let the coordinates of D be $D(x, y, z)$.

We know that midpoint of BD coincides with the midpoint of AC.

$$\therefore \quad \frac{2+x}{2}=\frac{4+1}{2}, \frac{3+y}{2}=\frac{5+2}{2} \text{ and } \frac{4+z}{2}=\frac{10+(-1)}{2}$$
$$\Rightarrow \quad 2+x=5,\ 3+y=7 \text{ and } 4+z=9$$
$$\Rightarrow \quad x=3,\ y=4 \text{ and } z=5.$$
$$\therefore \quad \text{coordinates of } D \text{ are } D(3, 4, 5).$$

EXERCISE 27A

1. A line passes through the point (3, 4, 5) and is parallel to the vector $(2\hat{i}+2\hat{j}-3\hat{k})$. Find the equations of the line in the vector as well as Cartesian forms.
2. A line passes through the point (2, 1, –3) and is parallel to the vector $(\hat{i}-2\hat{j}+3\hat{k})$. Find the equations of the line in vector and Cartesian forms.

3. Find the vector equation of the line passing through the point with position vector $(2\hat{i}+\hat{j}-5\hat{k})$ and parallel to the vector $(\hat{i}+3\hat{j}-\hat{k})$. Deduce the Cartesian equations of the line.

4. A line is drawn in the direction of $(\hat{i}+\hat{j}-2\hat{k})$ and it passes through a point with position vector $(2\hat{i}-\hat{j}-4\hat{k})$. Find the equations of the line in the vector as well as Cartesian forms.

5. The Cartesian equations of a line are

$$\frac{x-3}{2}=\frac{y+2}{-5}=\frac{z-6}{4}.$$

Find the vector equation of the line.

6. The Cartesian equations of a line are $3x+1=6y-2=1-z$. Find the fixed point through which it passes, its direction ratios and also its vector equation. **[CBSE 2004]**

7. Find the Cartesian equations of the line which passes through the point $(1, 3, -2)$ and is parallel to the line given by

$$\frac{x+1}{3}=\frac{y-4}{5}=\frac{z+3}{-6}.$$

Also, find the vector form of the equations so obtained.

8. Find the equations of the line passing through the point (1, –2, 3) and parallel to the line $\frac{x-6}{3}=\frac{y-2}{-4}=\frac{z+7}{5}$.

Also find the vector form of this equation so obtained.

9. Find the Cartesian and vector equations of a line which passes through the point (1, 2, 3) and is parallel to the line

$$\frac{-x-2}{1}=\frac{y+3}{7}=\frac{2z-6}{3}.$$ **[CBSE 2004, '07]**

10. Find the equations of the line passing through the point (–1, 3, –2) and perpendicular to each of the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and $\frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5}$.

[CBSE 2005, '12]

11. Find the Cartesian and vector equations of the line passing through the point (1, 2, –4) and perpendicular to each of the lines

$$\frac{x-8}{8}=\frac{y+19}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y+29}{8}=\frac{z-5}{-5}.$$ **[CBSE 2012]**

12. Prove that the lines

$$\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7} \text{ and } \frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$$

intersect each other and find the point of their intersection.

13. Show that the lines
$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-4}{5}=\frac{y-1}{2}=z$$ **[CBSE 2004, '05]**
intersect each other. Also, find the point of their intersection.

14. Show that the lines
$$\frac{x-1}{2}=\frac{y+1}{3}=z \text{ and } \frac{x+1}{5}=\frac{y-2}{1},\ z=2$$
do not intersect each other.

15. Find the coordinates of the foot of the perpendicular drawn from the point (1, 2, 3) to the line
$$\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}.$$ **[CBSE 2003C]**
Also, find the length of the perpendicular from the given point to the line.

16. Find the length and the foot of the perpendicular drawn from the point (2, −1, 5) to the line
$$\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}.$$ **[CBSE 2008, '09C]**

17. Find the vector and Cartesian equations of the line passing through the points $A(3, 4, -6)$ and $B(5, -2, 7)$.

18. Find the vector and Cartesian equations of the line passing through the points $A(2, -3, 0)$ and $B(-2, 4, 3)$.

19. Find the vector and Cartesian equations of the line joining the points whose position vectors are $(\hat{i}-2\hat{j}+\hat{k})$ and $(\hat{i}+3\hat{j}-2\hat{k})$.

20. Find the vector equation of a line passing through the point $A(3, -2, 1)$ and parallel to the line joining the points $B(-2, 4, 2)$ and $C(2, 3, 3)$. Also, find the Cartesian equations of the line.

21. Find the vector equation of a line passing through the point having the position vector $(\hat{i}+2\hat{j}-3\hat{k})$ and parallel to the line joining the points with position vectors $(\hat{i}-\hat{j}+5\hat{k})$ and $(2\hat{i}+3\hat{j}-4\hat{k})$. Also, find the Cartesian equivalents of this equation.

22. Find the coordinates of the foot of the perpendicular drawn from the point $A(1, 2, 1)$ to the line joining the points $B(1, 4, 6)$ and $C(5, 4, 4)$.

23. Find the coordinates of the foot of the perpendicular drawn from the point $A(1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$. **[CBSE 2005]**

24. Find the image of the point (0, 2, 3) in the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$.

25. Find the image of the point (5, 9, 3) in the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$.

26. Find the image of the point (2, –1, 5) in the line
$$\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k}).$$

ANSWERS (EXERCISE 27A)

1. $\vec{r} = (3\hat{i} + 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 2\hat{j} - 3\hat{k}),\ \frac{x-3}{2} = \frac{y-4}{2} = \frac{z-5}{-3}$

2. $\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 3\hat{k}),\ \frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+3}{3}$

3. $\vec{r} = (2\hat{i} + \hat{j} - 5\hat{k}) + \lambda(\hat{i} + 3\hat{j} - \hat{k}),\ \frac{x-2}{1} = \frac{y-1}{3} = \frac{z+5}{-1}$

4. $\vec{r} = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + \hat{j} - 2\hat{k}),\ \frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2}$

5. $\vec{r} = (3\hat{i} - 2\hat{j} + 6\hat{k}) + \lambda(2\hat{i} - 5\hat{j} + 4\hat{k})$

6. $\left(\frac{-1}{3}, \frac{1}{3}, 1\right)$, d.r.'s are $(2, 1, -6)$, $\vec{r} = \left(\frac{-1}{3}\hat{i} + \frac{1}{3}\hat{j} + \hat{k}\right) + \lambda(2\hat{i} + \hat{j} - 6\hat{k})$

7. $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z+2}{-6}$, $\vec{r} = (\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(3\hat{i} + 5\hat{j} - 6\hat{k})$

8. $\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z-3}{5}$, $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 4\hat{j} + 5\hat{k})$

9. $\frac{x-1}{-2} = \frac{y-2}{14} = \frac{z-3}{3}$, $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-2\hat{i} + 14\hat{j} + 3\hat{k})$

10. $\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$, $\vec{r} = (-\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k})$

11. $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$, $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

12. $(5, -7, 6)$ **13.** $(-1, -1, -1)$

15. $(3, 5, 9)$, 7 units **16.** $\sqrt{14}$ units, $(1, 2, 3)$

17. $\vec{r} = (3\hat{i} + 4\hat{j} - 6\hat{k}) + \lambda(2\hat{i} - 6\hat{j} + 13\hat{k}),\ \frac{x-3}{2} = \frac{y-4}{-6} = \frac{z+6}{13}$

18. $\vec{r} = (2\hat{i} - 3\hat{j}) + \lambda(-4\hat{i} + 7\hat{j} + 3\hat{k}),\ \frac{x-2}{-4} = \frac{y+3}{7} = \frac{z}{3}$

19. $\vec{r} = (\hat{i} - 2\hat{j} + \hat{k}) + \lambda(5\hat{j} - 3\hat{k}),\ \frac{x-1}{0} = \frac{y+2}{5} = \frac{z-1}{-3}$

20. $\vec{r} = (3\hat{i} - 2\hat{j} + \hat{k}) + \lambda(4\hat{i} - \hat{j} + \hat{k}),\ \frac{x-3}{4} = \frac{y+2}{1} = \frac{z-1}{1}$

21. $\vec{r} = (\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 4\hat{j} - 9\hat{k}),\ \frac{x-1}{1} = \frac{y-2}{4} = \frac{z+3}{-9}$

22. (3, 4, 5) **23.** $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$ **24.** (4, 4, –5)

25. (1, 1, 11) **26.** (0, 5, 1)

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 27A)

9. The given equation may be written as

$$\frac{x+2}{-1}=\frac{y+3}{7}=\frac{z-3}{\frac{3}{2}} \Rightarrow \frac{x+2}{-2}=\frac{y+3}{14}=\frac{z-3}{3}.$$

10. Let the required equations be $\frac{x+1}{a}=\frac{y-3}{b}=\frac{z+2}{c}$. Then,

$$a+2b+3c=0 \quad \text{... (i)}$$
$$-3a+2b+5c=0 \quad \text{... (ii)}$$

On solving (i) and (ii) by cross multiplication, we get

$$\frac{a}{(10-6)}=\frac{b}{(-9-5)}=\frac{c}{(2+6)} \Rightarrow \frac{a}{4}=\frac{b}{-14}=\frac{c}{8} \Rightarrow \frac{a}{2}=\frac{b}{-7}=\frac{c}{4}$$

Hence, the required equations are $\frac{x+1}{2}=\frac{y-3}{-7}=\frac{z+2}{4}$.

14. The second equation is $\frac{x+1}{5}=\frac{y-2}{1}=\frac{z-2}{0}$.

17. The given line passes through the point $A(3, 4, -6)$ and it has d.r.'s $(5-3), (-2-4)$, $(7+6)$, i.e., 2, –6, 13.

$\therefore$ its vector equation is $\vec{r}=(3\hat{i}+4\hat{j}-6\hat{k})+\lambda(2\hat{i}-6\hat{j}+13\hat{k})$.

Its Cartesian equations are $\frac{(x-3)}{2}=\frac{(y-4)}{-6}=\frac{(z+6)}{13}$.

19. The given line passes through the point $A(1, -2, 1)$ and it has d.r.'s $(1-1), (3+2)$, $(-2-1)$, i.e., 0, 5, –3.

So, its vector equation is $\vec{r}=(\hat{i}-2\hat{j}+\hat{k})+\lambda(5\hat{j}-3\hat{k})$.

And, its Cartesian equations are $\frac{x-1}{0}=\frac{y+2}{5}=\frac{z-1}{-3}$.

20. The given line passes through the point $A(3, -2, 1)$ and it has d.r.'s $(2+2), (3-4)$, $(3-2)$, i.e., 4, –1, 1.

21. The required line passes through the point, $A(1, 2, -3)$ and it is parallel to the line joining $B(1, -1, 5)$ and $C(2, 3, -4)$.

$\therefore$ its d.r.'s are $(2-1), (3+1), (-4-5)$, i.e., 1, 4, –9.

23. The equations of BC are $\frac{x-0}{0-2}=\frac{y+1}{-1+3}=\frac{z-3}{3+1} \Rightarrow \frac{x}{-2}=\frac{y+1}{2}=\frac{z-3}{4}$

$$\Rightarrow \frac{x}{-2}=\frac{y+1}{2}=\frac{z-3}{4} \Rightarrow \frac{x}{-1}=\frac{y+1}{1}=\frac{z-3}{2}=r \text{ (say)}.$$

Any general point on BC is $(-r, r-1, 2r+3)$.

Let $AP \perp BC$. Then $P(-r, r-1, 2r+3)$ for some fixed value of r.

d.r.'s of BC are –1, 1, 2.

d.r.'s of AP are $(-r-1),(r-1-8),(2r+3-4)$, i.e., $(-r-1),(r-9),(2r-1)$.

Since $AP \perp BC$, we have $-1\cdot(-r-1)+1\cdot(r-9)+2(2r-1)=0$.

$\therefore \quad 6r=10 \Rightarrow r=\frac{5}{3}$.

Hence, the coordinates of P are $\left(\frac{-5}{3}, \frac{5}{3}-1, \frac{10}{3}+3\right)$, i.e., $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$.

26. $x\hat{i}+y\hat{j}+z\hat{k}=(11+10\lambda)\hat{i}-(2+4\lambda)\hat{j}-(8+11\lambda)\hat{k}$

$\Leftrightarrow x=11+10\lambda, y=-(2+4\lambda)$ and $z=-(8+11\lambda)$

$\Leftrightarrow \frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}$ are the equations of the given line.

Collinearity of Three Given Points

When the coordinates of Three Points are Given

THEOREM 1 *The condition for three given points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ to be collinear is that*

$$\frac{x_3-x_1}{x_2-x_1}=\frac{y_3-y_1}{y_2-y_1}=\frac{z_3-z_1}{z_2-z_1}.$$

PROOF The equations of the line AB are

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}. \qquad \ldots \text{(i)}$$

Clearly, A, B, C will be collinear only when C lies on the line AB. This happens when $C(x_3, y_3, z_3)$ lies on (i).

$\therefore \quad \frac{x_3-x_1}{x_2-x_1}=\frac{y_3-y_1}{y_2-y_1}=\frac{z_3-z_1}{z_2-z_1}.$

When Position Vectors of Three Points are Given

THEOREM 2 *Three points A, B and C with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively are collinear if and only if there exist scalars d_1, d_2, d_3 not all zero such that*

$$d_1\vec{a}+d_2\vec{b}+d_3\vec{c}=\vec{0} \text{ and } d_1+d_2+d_3=0.$$

PROOF Let A, B and C be three collinear points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

Then, the vector equation of line $\overrightarrow{AB}$ is

$\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$ for some scalar λ

$\Rightarrow \quad \vec{c} = \vec{a} + \lambda(\vec{b} - \vec{a})$ [$\because A$, B and C being collinear, C lies on AB]

$\Rightarrow \quad (1-\lambda)\vec{a} + \lambda\vec{b} - \vec{c} = \vec{0}$

$\Rightarrow \quad d_1\vec{a} + d_2\vec{b} + d_3\vec{c} = \vec{0}$, where $d_3 = -1 \neq 0$ and

$$d_1 + d_2 + d_3 = (1-\lambda) + \lambda - 1 = 0.$$

Conversely, let A, B, and C be the given points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively and let d_1, d_2, d_3 be scalars, not all zero such that $d_1\vec{a} + d_2\vec{b} + d_3\vec{c} = \vec{0}$, and $d_1 + d_2 + d_3 = 0$.

Let $d_3 \neq 0$. Then,

$d_1\vec{a} + d_2\vec{b} + d_3\vec{c} = 0$, and $d_1 + d_2 + d_3 = 0$

$\Rightarrow \quad \left(\frac{d_1}{d_3}\right)\vec{a} + \left(\frac{d_2}{d_3}\right)\vec{b} + \vec{c} = \vec{0}$ and $\left(\frac{d_1}{d_3}\right) + \left(\frac{d_2}{d_3}\right) + 1 = 0$

$\Rightarrow \quad \left(\frac{d_1}{d_3}\right)\vec{a} - \lambda\vec{b} + \vec{c} = \vec{0}$ and $\left(\frac{d_1}{d_3}\right) - \lambda + 1 = 0$, where $\left(\frac{d_2}{d_3}\right) = -\lambda$

$\Rightarrow \quad (\lambda - 1)\vec{a} - \lambda\vec{b} + \vec{c} = \vec{0} \quad \left[\because \quad \left(\frac{d_1}{d_3}\right) = (\lambda - 1)\right]$

$\Rightarrow \quad \vec{c} = (1-\lambda)\vec{a} + \lambda\vec{b}$

$\Rightarrow$ the point C lies on the line $\overrightarrow{AB}$

$\Rightarrow$ the points A, B and C are collinear.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the points $A(2, 0, 3)$, $B(3, 2, -1)$ and $C(1, -2, -5)$ are collinear.*

SOLUTION The equations of the line AB are

$$\frac{x-2}{3-2} = \frac{y-0}{2-0} = \frac{z-3}{-1-3} \Rightarrow \frac{x-2}{1} = \frac{y}{2} = \frac{z-3}{-4}. \quad \text{... (i)}$$

Putting $x = 1$, $y = -2$ and $z = -5$ in (i), we get

$$\frac{1-2}{1} = \frac{-2}{2} = \frac{-5-3}{-4}, \text{ which is clearly true.}$$

Thus, the point $C(1, -2, -5)$ satisfies the equation of line AB.

$\therefore$ C lies on line AB.

Hence, the given points A, B and C are collinear.

EXAMPLE 2 *Find the value of λ for which the points $A(-1, 3, 2)$, $B(-4, 2, -2)$ and $C(5, 5, \lambda)$ are collinear.*

SOLUTION The equations of the line AB are

$$\frac{x+1}{-4+1} = \frac{y-3}{2-3} = \frac{z-2}{-2-2}$$

$$\Rightarrow \quad \frac{x+1}{-3} = \frac{y-3}{-1} = \frac{z-2}{-4}. \qquad \text{... (i)}$$

Since the points A, B and C are collinear, so the point $C(5, 5, \lambda)$ lies on (i).

$$\therefore \quad \frac{5+1}{-3} = \frac{5-3}{-1} = \frac{\lambda-2}{-4} \Rightarrow \frac{\lambda-2}{-4} = -2 \Rightarrow \lambda - 2 = 8 \Rightarrow \lambda = 10.$$

Hence, the required value of λ is 10.

EXAMPLE 3 *Show that the points whose position vectors are* $(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $(\hat{i} + 2\hat{j} + 3\hat{k})$ *and* $(7\hat{i} - \hat{k})$ *are collinear.*

SOLUTION The coordinates of the given points are $A(-2, 3, 5)$, $B(1, 2, 3)$ and $C(7, 0, -1)$.

The equations of line AB are

$$\frac{x+2}{1+2} = \frac{y-3}{2-3} = \frac{z-5}{3-5} \Rightarrow \frac{x+2}{3} = \frac{y-3}{-1} = \frac{z-5}{-2}. \qquad \text{... (i)}$$

Putting $x = 7$, $y = 0$ and $z = -1$ in (i), we get

$$\frac{7+2}{3} = \frac{0-3}{-1} = \frac{-1-5}{-2}, \text{ which is clearly true.}$$

Thus, the point $C(7, 0, -1)$ satisfies the equation of line AB.

$\therefore$ C lies on line AB.

Hence, the given points A, B and C are collinear.

EXAMPLE 4 *Using the vector method, find the values of* λ *and* μ *for which the points* $A(3, \lambda, \mu)$, $B(2, 0, -3)$ *and* $C(1, -2, -5)$ *are collinear.*

SOLUTION Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of the given points A, B and C respectively. Then,

$$\vec{a} = 3\hat{i} + \lambda\hat{j} + \mu\hat{k}, \quad \vec{b} = 2\hat{i} - 3\hat{k} \text{ and } \vec{c} = \hat{i} - 2\hat{j} - 5\hat{k}.$$

Now, the vector equation of the line BC is given by

$$\vec{r} = \vec{b} + \alpha(\vec{c} - \vec{b}) \text{ for some scalar } \alpha$$

$$\Rightarrow \quad \vec{r} = (1-\alpha)\,\vec{b} + \alpha\,\vec{c}$$

$$\Rightarrow \quad \vec{r} = (1-\alpha)\,(2\hat{i} - 3\hat{k}) + \alpha(\hat{i} - 2\hat{j} - 5\hat{k})$$

$$\Rightarrow \quad \vec{r} = (2 - 2\alpha + \alpha)\,\hat{i} - 2\alpha\,\hat{j} + (-3 + 3\alpha - 5\alpha)\,\hat{k}$$

$$\Rightarrow \quad \vec{r} = (2-\alpha)\,\hat{i} - 2\alpha\,\hat{j} + (-3 - 2\alpha)\,\hat{k}.$$

If this line passes through the point A then we must have

$$3\hat{i} + \lambda\hat{j} + \mu\hat{k} = (2-\alpha)\hat{i} - 2\alpha\hat{j} + (-3-2\alpha)\hat{k}$$

$\Leftrightarrow \quad 2-\alpha = 3, \; -2\alpha = \lambda$ and $-3-2\alpha = \mu$

$\Leftrightarrow \quad \alpha = -1, \; \lambda = 2$ and $\mu = (-3+2) = -1.$

Hence, $\lambda = 2$ and $\mu = -1.$

EXERCISE 27B

1. Show that the points $A(2, 1, 3)$, $B(5, 0, 5)$ and $C(-4, 3, -1)$ are collinear.
2. Show that the points $A(2, 3, -4)$, $B(1, -2, 3)$ and $C(3, 8, -11)$ are collinear. **[CBSE 2006]**
3. Find the value of λ for which the points $A(2, 5, 1)$, $B(1, 2, -1)$ and $C(3, \lambda, 3)$ are collinear.
4. Find the values of λ and μ so that the points $A(3, 2, -4)$, $B(9, 8, -10)$ and $C(\lambda, \mu, -6)$ are collinear.
5. Find the values of λ and μ so that the points $A(-1, 4, -2)$, $B(\lambda, \mu, 1)$ and $C(0, 2, -1)$ are collinear.
6. The position vectors of three points A, B and C are $(-4\hat{i} + 2\hat{j} - 3\hat{k})$, $(\hat{i} + 3\hat{j} - 2\hat{k})$ and $(-9\hat{i} + \hat{j} - 4\hat{k})$ respectively. Show that the points A, B and C are collinear.

ANSWERS (EXERCISE 27B)

3. $\lambda = 8$ **4.** $\lambda = 5, \; \mu = 4$ **5.** $\lambda = 2, \; \mu = -2$

Angle between Two Lines

(i) When the Given Lines are in Vector Form

Let the equations of the given lines in vector form be $\vec{r} = \vec{r_1} + \lambda\vec{m_1}$ and $\vec{r} = \vec{r_2} + \mu\vec{m_2}$, where λ and μ are scalars.

Let θ be the angle between these lines.

Since the given lines are parallel to $\vec{m_1}$ and $\vec{m_2}$ respectively, the angle between the given lines must be equal to the angle between $\vec{m_1}$ and $\vec{m_2}$.

$$\therefore \quad \cos\theta = \frac{|\vec{m_1} \cdot \vec{m_2}|}{|\vec{m_1}| \cdot |\vec{m_2}|}.$$

(ii) When the Given Lines are in Cartesian Form

Let the Cartesian equations of two given lines be

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}.$$

Then, the direction ratios of these lines are a_1, b_1, c_1, and a_2, b_2, c_2 respectively.

Let θ be the angle between these lines. Then,

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{(\sqrt{a_1^2 + b_1^2 + c_1^2})(\sqrt{a_2^2 + b_2^2 + c_2^2})}.$$

SUMMARY

(i) If θ is the angle between the lines $\vec{r} = \vec{r_1} + \lambda\vec{m_1}$ and $\vec{r} = \vec{r_2} + \mu\vec{m_2}$ then

$$\cos\theta = \frac{|\vec{m_1} \cdot \vec{m_2}|}{|\vec{m_1}| \cdot |\vec{m_2}|}.$$

REMARK: $\theta = \frac{\pi}{2} \Leftrightarrow m_1 \cdot m_2 = 0.$

(ii) If θ is the angle between the lines whose Cartesian equations are

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \text{ then}$$

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{(\sqrt{a_1^2 + b_1^2 + c_1^2})(\sqrt{a_2^2 + b_2^2 + c_2^2})}.$$

REMARK: $\theta = \frac{\pi}{2} \Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0.$

SOLVED EXAMPLES

EXAMPLE 1 *Find the angle between the lines*

$$\vec{r} = (2\hat{i} - 5\hat{j} + \hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k}) \text{ and}$$
$$\vec{r} = (7\hat{i} - 6\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k}).$$ **[CBSE 2014]**

SOLUTION The given lines are of the form

$$\vec{r} = \vec{r_1} + \lambda\vec{m_1} \text{ and } \vec{r} = \vec{r_2} + \lambda\vec{m_2}, \text{ where}$$
$$\vec{m_1} = (3\hat{i} + 2\hat{j} + 6\hat{k}) \text{ and } \vec{m_2} = (\hat{i} + 2\hat{j} + 2\hat{k}).$$

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{|\vec{m_1}\cdot\vec{m_2}|}{|\vec{m_1}|\,|\vec{m_2}|}$$

$$= \frac{|(3\hat{i}+2\hat{j}+6\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})|}{(\sqrt{3^2+2^2+6^2})\cdot(\sqrt{1^2+2^2+2^2})} = \frac{(3+4+12)}{(\sqrt{49})(\sqrt{9})}$$

$$= \frac{19}{(7\times 3)} = \frac{19}{21}.$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{19}{21}\right).$$

Hence, the angle between the given lines is $\cos^{-1}\left(\frac{19}{21}\right)$.

EXAMPLE 2 *Find the angle between the lines*

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}.$$ **[CBSE 2011]**

SOLUTION The given equations in standard form are

$$\frac{x-2}{2} = \frac{y-1}{7} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{2} = \frac{z-5}{4}.$$

Here $a_1 = 2,\ b_1 = 7,\ c_1 = -3$ and $a_2 = -1,\ b_2 = 2,\ c_2 = 4.$

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{(\sqrt{a_1^2+b_1^2+c_1^2})(\sqrt{a_2^2+b_2^2+c_2^2})}$$

$$= \frac{|2\times(-1)+7\times 2+(-3)\times 4|}{\{\sqrt{2^2+7^2+(-3)^2}\}\{\sqrt{(-1)^2+2^2+4^2}\}}$$

$$= \frac{|(-2)+14+(-12)|}{(\sqrt{62})(\sqrt{21})} = 0.$$

$$\therefore \quad \theta = \frac{\pi}{2}.$$

Hence, the angle between the given lines is $\frac{\pi}{2}$.

EXAMPLE 3 *Find the value of λ so that the following lines are perpendicular to each other:*

$$\frac{x-5}{5\lambda+2} = \frac{2-y}{5} = \frac{1-z}{-1} \text{ and } \frac{x}{1} = \frac{2y+1}{4\lambda} = \frac{1-z}{-3}.$$ **[CBSE 2009]**

SOLUTION The given equations in standard form are

$$\frac{x-5}{5\lambda+2} = \frac{y-2}{-5} = \frac{z-1}{1} \text{ and } \frac{x}{1} = \frac{y+\frac{1}{2}}{2\lambda} = \frac{z-1}{3}.$$

Here $a_1 = 5\lambda + 2,\ b_1 = -5,\ c_1 = 1$ and $a_2 = 1,\ b_2 = 2\lambda$ and $c_2 = 3$.

Since the given lines are perpendicular to each other, we must have $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

$\therefore \quad (5\lambda + 2) \times 1 + (-5) \times (2\lambda) + 1 \times 3 = 0$

$\Rightarrow \quad 5\lambda + 2 - 10\lambda + 3 = 0 \ \Rightarrow \ 5\lambda = 5 \ \Rightarrow \ \lambda = 1.$

Hence, the required value of λ is 1.

EXAMPLE 4 *Find the angle between the lines*

$$\frac{x+1}{1} = \frac{2y-3}{3} = \frac{z-6}{2} \text{ and } \frac{x-4}{3} = \frac{y+3}{-2},\ z = 5.$$

SOLUTION The given equations may be written as

$$\frac{x+1}{1} = \frac{y-\frac{3}{2}}{\frac{3}{2}} = \frac{z-6}{2} \Rightarrow \frac{x+1}{2} = \frac{y-\frac{3}{2}}{3} = \frac{z-6}{4} \qquad \ldots \text{(i)}$$

and $$\frac{x-4}{3} = \frac{y+3}{-2} = \frac{z-5}{0}. \qquad \ldots \text{(ii)}$$

Here $a_1 = 2,\ b_1 = 3,\ c_1 = 4$ and $a_2 = 3,\ b_2 = -2, c_2 = 0$.

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{(\sqrt{a_1^2 + b_1^2 + c_1^2})(\sqrt{a_2^2 + b_2^2 + c_2^2})}$$

$$= \frac{|(2\times 3) + 3\times(-2) + 4\times 0|}{(\sqrt{4+9+16})(\sqrt{9+4+0})} = \frac{0}{(\sqrt{29}\times\sqrt{13})} = 0.$$

$\therefore \quad \theta = \frac{\pi}{2}.$

Hence, the angle between the given lines is $\frac{\pi}{2}$.

EXAMPLE 5 *Find the angle between the lines*

$$\frac{5-x}{3} = \frac{y+3}{-4},\ z = 7 \text{ and } \frac{x}{1} = \frac{1-y}{2} = \frac{z-6}{2}.$$

SOLUTION The given lines in standard form are

$$\frac{x-5}{-3} = \frac{y+3}{-4} = \frac{z-7}{0} \text{ and } \frac{x}{1} = \frac{y-1}{-2} = \frac{z-6}{2}$$

Here $a_1 = -3,\ b_1 = -4,\ c_1 = 0$ and $a_2 = 1,\ b_2 = -2,\ c_2 = 2$.

Let θ be the angle between the given lines. Then,

$$\cos 0 = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\{\sqrt{a_1^2 + b_1^2 + c_1^2}\}\{\sqrt{a_2^2 + b_2^2 + c_2^2}\}}$$

$$= \frac{|(-3)\times 1 + (-4)\times(-2) + 0\times 2|}{\{\sqrt{9+16+0}\}\{\sqrt{1+4+4}\}} = \frac{5}{5\times 3} = \frac{1}{3}.$$

$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$.

Hence, the angle between the given lines is $\cos^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 6 *Find the value of k so that the lines*

$$\frac{1-x}{3} = \frac{7y-14}{2k} = \frac{z-3}{2} \text{ and } \frac{7-7x}{3k} = \frac{5-y}{1} = \frac{6-z}{5}$$

are at right angles.

SOLUTION The given equation in standard form are

$$\frac{x-1}{-3} = \frac{y-2}{\left(\frac{2k}{7}\right)} = \frac{z-3}{2} \Rightarrow \frac{x-1}{-21} = \frac{y-2}{2k} = \frac{z-3}{14} \quad \ldots \text{(i)}$$

$$\frac{x-1}{\left(\frac{-3k}{7}\right)} = \frac{y-5}{-1} = \frac{z-6}{-5} \Rightarrow \frac{x-1}{-3k} = \frac{y-5}{-7} = \frac{z-6}{-35}. \quad \ldots \text{(ii)}$$

Here, $a_1 = -21$, $b_1 = 2k$, $c_1 = 14$ and $a_2 = -3k$, $b_2 = -7$, $c_2 = -35$.

Since the given lines are at right angles, we have

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow \quad (-21)\times(-3k) + (2k)\times(-7) + 14\times(-35) = 0$$

$$\Rightarrow \quad 63k - 14k - 490 = 0 \Rightarrow 49k = 490 \Rightarrow k = 10.$$

Hence, $k = 10$.

EXAMPLE 7 *Prove that the lines* $x = ay + b, z = cy + d$, *and* $x = a'y + b', z = c'y + d'$ *are perpendicular if* $aa' + cc' + 1 = 0$.

SOLUTION The equations of the first line are

$$x = ay + b, z = cy + d$$

$$\Leftrightarrow \quad \frac{x-b}{a} = y, \frac{z-d}{c} = y$$

$$\Leftrightarrow \quad \frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c}. \quad \ldots \text{(i)}$$

Similarly, the equations of the second line are

$$\frac{x-b'}{a'} = \frac{y}{1} = \frac{z-d'}{c'}. \quad \ldots \text{(ii)}$$

The given lines are perpendicular to each other

$$\Leftrightarrow \quad aa' + 1\times 1 + cc' = 0$$

$$\Leftrightarrow \quad aa' + cc' + 1 = 0.$$

EXAMPLE 8 *Find the angle between the two lines, one of which has direction ratios* 2, 2, 1, *and the other is obtained by joining the points* (3, 1, 4) *and* (7, 2, 12).

SOLUTION Let L_1 and L_2 be the given lines.

Then, d.r.'s of L_1 are 2, 2, 1.

D.r.'s of L_2 are $(7-3), (2-1), (12-4)$, i.e., 4, 1, 8.

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{|(2\times 4 + 2\times 1 + 1\times 8)|}{(\sqrt{2^2+2^2+1^2})\,(\sqrt{4^2+1^2+8^2})} = \frac{18}{27} = \frac{2}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{3}\right).$$

Hence, the angle between the given lines is $\cos^{-1}\left(\frac{2}{3}\right)$.

EXERCISE 27C

Find the angle between each of the following pairs of lines:

1. $\vec{r} = (3\hat{i} + \hat{j} - 2\hat{k}) + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - 5\hat{k}) + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$
2. $\vec{r} = (3\hat{i} - 4\hat{j} + 2\hat{k}) + \lambda(\hat{i} + 3\hat{k})$ and $\vec{r} = 5\hat{i} + \mu(-\hat{i} + \hat{j} + \hat{k})$
3. $\vec{r} = (\hat{i} - 2\hat{j}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r} = 3\hat{k} + \mu(\hat{i} + 2\hat{j} - 2\hat{k})$

Find the angle between each of the following pairs of lines:

4. $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$ and $\frac{x+3}{3} = \frac{y-2}{5} = \frac{z+5}{4}$
5. $\frac{x-4}{3} = \frac{y+1}{4} = \frac{z-6}{5}$ and $\frac{x-5}{1} = \frac{2y+5}{-2} = \frac{z-3}{1}$
6. $\frac{3-x}{-2} = \frac{y+5}{1} = \frac{1-z}{3}$ and $\frac{x}{3} = \frac{1-y}{-2} = \frac{z+2}{-1}$
7. $\frac{x}{1} = \frac{z}{-1}$, $y = 0$ and $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$
8. $\frac{5-x}{3} = \frac{y+3}{-2}$, $z = 5$ and $\frac{x-1}{1} = \frac{1-y}{3} = \frac{z-5}{2}$
9. Show that the lines $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z-2}{4}$ and $\frac{x+2}{2} = \frac{y-4}{4} = \frac{z+5}{2}$ are perpendicular to each other.
10. If the lines $\frac{x-1}{-3} = \frac{y-2}{2\lambda} = \frac{z-3}{2}$ and $\frac{x-1}{3\lambda} = \frac{y-1}{1} = \frac{6-z}{5}$ are perpendicular to each other then find the value of λ. **[CBSE 2009]**
11. Show that the lines $x = -y = 2z$ and $x + 2 = 2y - 1 = -z + 1$ are perpendicular to each other.

HINT: The given lines are $\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$ and $\frac{x+2}{2} = \frac{y-1/2}{1} = \frac{z-1}{-2}$.

12. Find the angle between two lines whose direction ratios are
(i) 2, 1, 2 and 4, 8, 1 (ii) 5, –12, 13 and –3, 4, 5
(iii) 1, 1, 2 and $(\sqrt{3}-1), (-\sqrt{3}-1), 4$ (iv) a, b, c and $(b-c), (c-a), (a-b)$

13. If $A(1, 2, 3)$, $B(4, 5, 7)$, $C(-4, 3, -6)$ and $D(2, 9, 2)$ are four given points then find the angle between the lines AB and CD.

ANSWERS (EXERCISE 27C)

1. $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$ **2.** $\cos^{-1}\left(\frac{\sqrt{30}}{15}\right)$ **3.** $\cos^{-1}\left(\frac{-4}{9}\right)$ **4.** $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$

5. $\cos^{-1}\left(\frac{2\sqrt{6}}{15}\right)$ **6.** $\cos^{-1}\left(\frac{11}{14}\right)$ **7.** $\cos^{-1}\left(\frac{1}{5}\right)$ **8.** $\cos^{-1}\left(\frac{3}{\sqrt{182}}\right)$

10. $\lambda = \frac{-10}{7}$ **12.** (i) $\cos^{-1}\left(\frac{2}{3}\right)$ (ii) $\cos^{-1}\left(\frac{1}{65}\right)$ (iii) $\frac{\pi}{3}$ (iv) $\frac{\pi}{2}$ **13.** 0°

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 27C)

5. Given lines in standard form are:

$$\frac{x-4}{3}=\frac{y+1}{4}=\frac{z-6}{5} \text{ and } \frac{x-5}{1}=\frac{y+\frac{5}{2}}{-1}=\frac{z-3}{1}.$$

6. Given lines in standard form are:

$$\frac{x-3}{2}=\frac{y+5}{1}=\frac{z-1}{-3} \text{ and } \frac{x}{3}=\frac{y-1}{2}=\frac{z+2}{-1}.$$

7. Given lines in standard form are:

$$\frac{x}{1}=\frac{y-0}{0}=\frac{z}{-1} \text{ and } \frac{x}{3}=\frac{y}{4}=\frac{z}{5}.$$

8. Given lines in standard form are:

$$\frac{x-5}{-3}=\frac{y+3}{-2}=\frac{z-5}{0} \text{ and } \frac{x-1}{1}=\frac{y-1}{-3}=\frac{z-5}{2}.$$

9. Given lines in standard form are:

$$\frac{x}{2}=\frac{y}{-2}=\frac{z}{1} \text{ and } \frac{x+2}{2}=\frac{y-\frac{1}{2}}{1}=\frac{z-1}{-2}.$$

13. $\overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A) = (4\hat{i}+5\hat{j}+7\hat{k}) - (\hat{i}+2\hat{j}+3\hat{k}) = (3\hat{i}+3\hat{j}+4\hat{k})$

$\overrightarrow{CD} = (\text{p.v. of } D) - (\text{p.v. of } C) = (2\hat{i}+9\hat{j}+2\hat{k}) - (-4\hat{i}+3\hat{j}-6\hat{k}) = (6\hat{i}+6\hat{j}+8\hat{k})$

$$\cos\theta = \frac{|\overrightarrow{AB}\cdot\overrightarrow{CD}|}{|\overrightarrow{AB}||\overrightarrow{CD}|} = \frac{(3\times 6+3\times 6+4\times 8)}{(\sqrt{9+9+16})(\sqrt{36+36+64})}$$

$$= \frac{68}{\sqrt{34}\times\sqrt{136}} = \frac{68}{34\times 2} = 1.$$

Shortest Distance between Two Lines

COPLANAR LINES *Two lines lying in the same plane are called coplanar lines.*

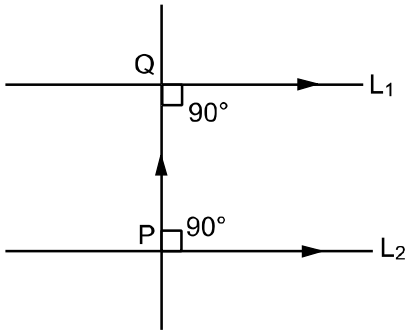
Coplanar lines are either parallel or intersecting.

SKEW LINES *Two lines in space which are not coplanar are called skew lines.*

Skew lines are neither parallel nor intersecting.

LINE OF SHORTEST DISTANCE BETWEEN TWO SKEW LINES If L_1 and L_2 are two skew lines then there is a unique line which is perpendicular to both the lines L_1 and L_2. This line is called the *line of shortest distance* between L_1 and L_2.

SHORTEST DISTANCE BETWEEN TWO SKEW LINES The length of the line segment $\overrightarrow{PQ}$, intercepted by two skew lines L_1 and L_2 on the common perpendicular to both the lines, is called the *shortest distance* (SD) between L_1 and L_2.



REMARK If two lines in space intersect at a point then the shortest distance between them is zero.

To Find the Shortest Distance between Two Skew Lines

Vector Form

THEOREM *The shortest distance between two skew lines* $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ *and* $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ *is given by*

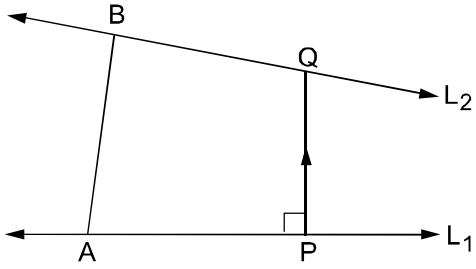
$$d = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|.$$

PROOF Let L_1 and L_2 be two skew lines whose vector equations are respectively

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1} \qquad \text{... (i)}$$

and $$\vec{r} = \vec{a_2} + \mu \vec{b_2}. \qquad \text{... (ii)}$$

Then, L_1 is parallel to $\vec{b_1}$ and passes through a point A, whose position vector is $\vec{a_1}$.



And, L_2 is parallel to $\vec{b_2}$ and passes through a point B, whose position vector is $\vec{a_2}$.

Let $\overrightarrow{PQ}$ be the shortest-distance vector between L_1 and L_2.

Then, $\overrightarrow{PQ} \perp \vec{b_1}$ and $\overrightarrow{PQ} \perp \vec{b_2}$.

$\therefore \quad \overrightarrow{PQ} \,||\, (\vec{b_1} \times \vec{b_2})$.

$\therefore \quad PQ = |\text{ projection of } \overrightarrow{AB} \text{ along } (\vec{b_1} \times \vec{b_2})|$

$$= \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|.$$

CONDITION FOR TWO GIVEN LINES TO INTERSECT Suppose that the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ intersect. Then, the shortest distance between them is zero.

$\therefore \quad [(\vec{a_2} - \vec{a_1}) \ \vec{b_1} \ \vec{b_2}] = 0.$

REMARK Two lines intersect only when the shortest distance between them is zero.

SOLVED EXAMPLES

EXAMPLE 1 *Find the shortest distance between the lines whose vector equations are*

$$\vec{r} = (6\hat{i} + 2\hat{j} + 2\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$$

and $\vec{r} = (-4\hat{i} - \hat{k}) + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$. **[CBSE 2013C]**

SOLUTION Comparing the given equations with the standard equations $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$, we have

$$\vec{a_1} = (6\hat{i} + 2\hat{j} + 2\hat{k}), \ \vec{b_1} = (\hat{i} - 2\hat{j} + 2\hat{k}),$$

$$\vec{a_2} = (-4\hat{i} - \hat{k}) \text{ and } \vec{b_2} = (3\hat{i} - 2\hat{j} - 2\hat{k}).$$

$$\therefore \quad (\vec{a_2} - \vec{a_1}) = (-4\hat{i} - \hat{k}) - (6\hat{i} + 2\hat{j} + 2\hat{k}) = (-10\hat{i} - 2\hat{j} - 3\hat{k})$$

$$\text{and, } (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{i} - (-2-6)\hat{j} + (-2+6)\hat{k}$$

$$= (8\hat{i} + 8\hat{j} + 4\hat{k}).$$

$$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{8^2 + 8^2 + 4^2} = \sqrt{64 + 64 + 16}$$
$$= \sqrt{144} = 12.$$
$$\therefore \quad \text{SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$
$$= \left| \frac{(-10\hat{i} - 2\hat{j} - 3\hat{k}) \cdot (8\hat{i} + 8\hat{j} + 4\hat{k})}{12} \right| = \left| \frac{-80 - 16 - 12}{12} \right|$$
$$= \left| \frac{-108}{12} \right| = |-9| = 9 \text{ units.}$$

EXAMPLE 2 *Find the shortest distance between the lines L_1 and L_2 whose vector equations are given below:*

$$L_1 : \vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$$
$$L_2 : \vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{i} + 2\hat{k})$$

[CBSE 2008C, '14]

SOLUTION Comparing the given equations with the standard equations $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$, we have

$$\vec{a_1} = (\hat{i} + \hat{j}), \quad \vec{b_1} = (2\hat{i} - \hat{j} + \hat{k})$$
$$\vec{a_2} = (2\hat{i} + \hat{j} - \hat{k}) \quad \text{and} \quad \vec{b_2} = (3\hat{i} - 5\hat{j} + 2\hat{k})$$
$$\therefore \quad (\vec{a_2} - \vec{a_1}) = (2\hat{i} + \hat{j} - \hat{k}) - (\hat{i} + \hat{j}) = (\hat{i} - \hat{k})$$
$$\text{and, } (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = (-2 + 5)\hat{i} - (4 - 3)\hat{j} + (-10 + 3)\hat{k}$$
$$= (3\hat{i} - \hat{j} - 7\hat{k}).$$
$$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{3^2 + (-1)^2 + (-7)^2} = \sqrt{59}.$$
$$\therefore \quad \text{SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| = \frac{|(\hat{i} - \hat{k}) \cdot (3\hat{i} - \hat{j} - 7\hat{k})|}{\sqrt{59}}$$
$$= \frac{|3 - 0 + 7|}{\sqrt{59}} = \frac{10\sqrt{59}}{59} \text{ units.}$$

EXAMPLE 3 *Find the shortest distance between the lines whose vector equations are*

$$\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}, \text{ and}$$
$$\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}.$$

[CBSE 2011]

SOLUTION The given equations can be written as

$$\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}), \text{ and}$$

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}).$$

Comparing the given equations with the standard equations $\vec{r} = \vec{a_1} + t\,\vec{b_1}$ and $\vec{r} = \vec{a_2} + s\,\vec{b_2}$, we get

$$\vec{a_1} = (\hat{i} - 2\hat{j} + 3\hat{k}), \quad \vec{b_1} = (-\hat{i} + \hat{j} - 2\hat{k})$$

$$\vec{a_2} = (\hat{i} - \hat{j} - \hat{k}) \text{ and } \vec{b_2} = (\hat{i} + 2\hat{j} - 2\hat{k}).$$

$$\therefore \quad (\vec{a_2} - \vec{a_1}) = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = (\hat{j} - 4\hat{k})$$

$$\text{and, } (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= (-2 + 4)\hat{i} - (2 + 2)\hat{j} + (-2 - 1)\hat{k}$$

$$= (2\hat{i} - 4\hat{j} - 3\hat{k}).$$

$$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}.$$

$$\therefore \quad \text{SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \frac{|(\hat{j} - 4\hat{k}) \cdot (2\hat{i} - 4\hat{j} - 3\hat{k})|}{\sqrt{29}} = \frac{|0 - 4 + 12|}{\sqrt{29}}$$

$$= \frac{8\sqrt{29}}{29} \text{ units.}$$

EXAMPLE 4 *Show that the lines*

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) \text{ and } \vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

intersect. Find their point of intersection. **[CBSE 2014]**

SOLUTION Comparing the given equations with the standard equations $\vec{r} = \vec{a_1} + \lambda\,\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\,\vec{b_2}$, we get

$$\vec{a_1} = (\hat{i} + \hat{j} - \hat{k}), \quad \vec{b_1} = (3\hat{i} - \hat{j}),$$

$$\vec{a_2} = (4\hat{i} - \hat{k}) \text{ and } \vec{b_2} = (2\hat{i} + 3\hat{k}).$$

$$\therefore \quad (\vec{a_2} - \vec{a_1}) = (4\hat{i} - \hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = (3\hat{i} - \hat{j}).$$

$$\text{And, } (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = (-3-0)\hat{i} - (9-0)\hat{j} + (0+2)\hat{k}$$

$$= (-3\hat{i} - 9\hat{j} + 2\hat{k}).$$

$$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-3)^2 + (-9)^2 + 2^2} = \sqrt{94}.$$

$$\therefore \quad \text{SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \frac{|(3\hat{i} - \hat{j}) \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k})|}{\sqrt{94}}$$

$$= \frac{|-9+9+0|}{\sqrt{94}} = 0.$$

Thus, the shortest distance between the given lines is 0.

Hence, the given lines intersect.

Thus, for some particular values of λ and μ, we have

$$(\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

$$\Rightarrow \quad (1+3\lambda)\hat{i} + (1-\lambda)\hat{j} - \hat{k} = (4+2\mu)\hat{i} + (3\mu - 1)\hat{k}$$

$$\Rightarrow \quad 1 + 3\lambda = 4 + 2\mu, \; 1 - \lambda = 0 \text{ and } 3\mu - 1 = -1$$

$$\Rightarrow \quad \lambda = 1 \text{ and } \mu = 0.$$

Thus, the position vector of the point of intersection of the given lines is given by

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + (3\hat{i} - \hat{j}) \quad [\text{putting } \lambda = 1], \text{ i.e., } \vec{r} = (4\hat{i} - \hat{k}).$$

Hence, the point of intersection of the given lines is $P(4, 0, -1)$.

EXAMPLE 5 *Show that the lines*

$$\vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{i} + \hat{k}) \text{ and } \vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} + \hat{j} - \hat{k})$$

do not intersect. **[CBSE 2012C]**

SOLUTION Comparing the given equations with the standard equations $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$, we get

$$\vec{a_1} = (\hat{i} - \hat{j}), \; \vec{b_1} = (2\hat{i} + \hat{k})$$

$$\vec{a_2} = (2\hat{i} - \hat{j}) \text{ and } \vec{b_2} = (\hat{i} + \hat{j} - \hat{k}).$$

$$\therefore \quad (\vec{a_2} - \vec{a_1}) = (2\hat{i} - \hat{j}) - (\hat{i} - \hat{j}) = \hat{i}.$$

$$\text{And, } (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = (0-1)\hat{i} - (-2-1)\hat{j} + (2-0)\hat{k}$$

$$= (-\hat{i} + 3\hat{j} + 2\hat{k}).$$

$$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}.$$

$$\therefore \quad \text{SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \left| \frac{\hat{i} \cdot (-\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{14}} \right| = \frac{|-1|}{\sqrt{14}}$$

$$= \frac{1 \times \sqrt{14}}{14} = \frac{\sqrt{14}}{14} \neq 0.$$

Since the shortest distance between the given lines is not zero, the given lines do not intersect.

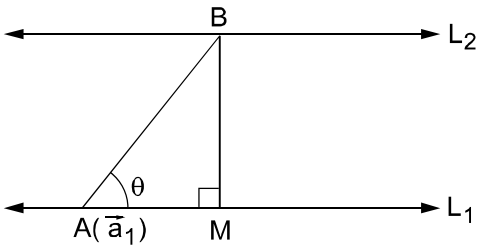
DISTANCE BETWEEN PARALLEL LINES Let L_1 and L_2 be two parallel lines. Then, these lines are clearly coplanar.

Let the equations of these lines be

$$\vec{r} = \vec{a_1} + \lambda\, \vec{b} \qquad \text{... (i)}$$

$$\vec{r} = \vec{a_2} + \mu\, \vec{b}. \qquad \text{... (ii)}$$

Let A be a point on L_1 with position vector $\vec{a_1}$ and let B be a point on L_2 with position vector $\vec{a_2}$.



Draw $BM \perp L_1$. Then,

distance between L_1 and $L_2 = |\overrightarrow{BM}|$.

Let θ be the angle between $\overrightarrow{AB}$ and $\vec{b}$. Then,

$$(\vec{b} \times \overrightarrow{AB}) = \{|\vec{b}|\,|\overrightarrow{AB}| \cdot \sin\theta\}\hat{n},$$

where $\hat{n}$ is a unit vector, perpendicular to the plane of L_1 and L_2.

$$\therefore \quad \vec{b} \times (\vec{a_2} - \vec{a_1}) = \{|\vec{b}|\,|\overrightarrow{AB}| \cdot \sin\theta\}\,\hat{n}$$

$$\Rightarrow \quad \vec{b} \times (\vec{a_2} - \vec{a_1}) = |\vec{b}|(BM)\,\vec{n} \qquad \{\because \ (AB)\sin\theta = BM\}$$

$$\Rightarrow \quad |\vec{b} \times (\vec{a_2} - \vec{a_1})| = |\vec{b}|\,|\overrightarrow{BM}| \cdot 1 \qquad [\because \ |\hat{n}| = 1]$$

$$\Rightarrow \quad |\overrightarrow{BM}| = \left| \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right|.$$

EXAMPLE 6 *Find the shortest distance between the lines L_1 and L_2, given by* $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ *and* $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(4\hat{i} - 2\hat{j} + 2\hat{k})$.

SOLUTION The given lines are

$$L_1 : \vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k}) \qquad \text{... (i)}$$

$$L_2 : \vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + 2\mu(2\hat{i} - \hat{j} + \hat{k}). \qquad \text{... (ii)}$$

These equations are of the form:

$\vec{r} = \vec{a_1} + \lambda \vec{b}$ and $\vec{r} = \vec{a_2} + 2\mu \ \vec{b} = \vec{a_2} + \mu' \ \vec{b}$, where

$\vec{a_1} = (\hat{i} + \hat{j}), \vec{a_2} = (2\hat{i} + \hat{j} - \hat{k}), \ \vec{b} = (2\hat{i} - \hat{j} + \hat{k})$ and $\mu' = 2\mu$.

Clearly, the given lines are parallel.

Now, $(\vec{a_2} - \vec{a_1}) = (2\hat{i} + \hat{j} - \hat{k}) - (\hat{i} + \hat{j}) = (\hat{i} - \hat{k})$

$$\therefore \quad \{\vec{b} \times (\vec{a_2} - \vec{a_1})\} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= (1 - 0)\hat{i} - (-2 - 1)\hat{j} + (0 + 1)\hat{k}$$

$$= (\hat{i} + 3\hat{j} + \hat{k})$$

$$\Rightarrow \quad |\vec{b} \times (\vec{a_2} - \vec{a_1})| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$$

and $|\vec{b}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$

$\Rightarrow$ shortest distance between L_1 and L_2

= distance between L_1 and L_2

$$= \frac{|\vec{b} \times (\vec{a_2} - \vec{a_1})|}{|\vec{b}|} = \frac{\sqrt{11}}{\sqrt{6}}$$

$$= \left(\frac{\sqrt{11}}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} \right) = \frac{\sqrt{66}}{6} \text{ units.}$$

EXAMPLE 7 *Write the vector equations of the following lines and hence find the shortest distance between them:*

$$\frac{x+1}{2} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$ **[CBSE 2008, '14]**

SOLUTION The given lines are

$$\frac{x+1}{2}=\frac{y+1}{-6}=\frac{z+1}{1} \quad \text{... (i)}$$

and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$. ... (ii)

The line (i) passes through the point (–1, –1, –1) and has d.r.'s. 7, –6, 1.

So, its vector equation is

$$\vec{r}=\vec{a_1}+\lambda\vec{b_1}, \quad \text{... (iii)}$$

where $\vec{a_1}=(-\hat{i}-\hat{j}-\hat{k})$ and $\vec{b_1}=7\hat{i}-6\hat{j}+\hat{k}$.

The line (ii) passes through the point (3, 5, 7) and has d.r.'s. 1, –2, 1.

So, its vector equation is

$$\vec{r}=\vec{a_2}+\lambda\vec{b_2}, \quad \text{... (iv)}$$

where $\vec{a_2}=(3\hat{i}+5\hat{j}+7\hat{k})$ and $\vec{b_2}=\hat{i}-2\hat{j}+\hat{k}$.

$$\therefore \quad (\vec{a_2}-\vec{a_1})=(3\hat{i}+5\hat{j}+7\hat{k})-(-\hat{i}-\hat{j}-\hat{k})=(4\hat{i}+6\hat{j}+8\hat{k})$$

$$\text{and } (\vec{b_1}\times\vec{b_2})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 7 & -6 & 1\\ 1 & -2 & 1\end{vmatrix}=(-6+2)\hat{i}-(7-1)\hat{j}+(-14+6)\hat{k}$$

$$=(-4\hat{i}-6\hat{j}-8\hat{k}).$$

$$\therefore \quad |\vec{b_1}\times\vec{b_2}|=\sqrt{(-4)^2+(-6)^2+(-8)^2}=\sqrt{16+36+64}=\sqrt{116}.$$

$$\therefore \quad \text{SD}=\left|\frac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|=\left|\frac{(4\hat{i}+6\hat{j}+8\hat{k})\cdot(-4\hat{i}-6\hat{j}-8\hat{k})}{\sqrt{116}}\right|$$

$$=\frac{|(-16-36-64)|}{\sqrt{116}}=\frac{|-116|}{\sqrt{116}}=\frac{116}{\sqrt{116}}\times\frac{\sqrt{116}}{\sqrt{116}}$$

$$=\sqrt{116}=\sqrt{4\times29}=2\sqrt{29} \text{ units.}$$

EXERCISE 27D

In problems* 1–8, *find the shortest distance between the given lines.

1. $\vec{r}=(\hat{i}+\hat{j})+\lambda(2\hat{i}-\hat{j}+\hat{k})$,

   $\vec{r}=(2\hat{i}+\hat{j}-\hat{k})+\mu(3\hat{i}-5\hat{j}+2\hat{k})$. **[CBSE 2007, '08C, '09, '14]**

2. $\vec{r} = (-4\hat{i} + 4\hat{j} + \hat{k}) + \lambda(\hat{i} + \hat{j} - \hat{k})$,

$\vec{r} = (-3\hat{i} - 8\hat{j} - 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 3\hat{k})$.

3. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$,

$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$. **[CBSE 2006, '11]**

4. $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$,

$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$. **[CBSE 2008, '11]**

5. $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$,

$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(-2\hat{i} + 3\hat{j} + 8\hat{k})$.

6. $\vec{r} = (6\hat{i} + 3\hat{k}) + \lambda(2\hat{i} - \hat{j} + 4\hat{k})$,

$\vec{r} = (-9\hat{i} + \hat{j} - 10\hat{k}) + \mu(4\hat{i} + \hat{j} + 6\hat{k})$.

7. $\vec{r} = (3 - t)\hat{i} + (4 + 2t)\hat{j} + (t - 2)\hat{k}$,

$\vec{r} = (1 + s)\hat{i} + (3s - 7)\hat{j} + (2s - 2)\hat{k}$.

8. $\vec{r} = (\lambda - 1)\hat{i} + (\lambda + 1)\hat{j} - (\lambda + 1)\hat{k}$,

$\vec{r} = (1 - \mu)\hat{i} + (2\mu - 1)\hat{j} + (\mu + 2)\hat{k}$.

9. Compute the shortest distance between the lines

$\vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{i} - \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} - \hat{j} - \hat{k})$.

Determine whether these lines intersect or not. **[CBSE 2012C]**

10. Show that the lines

$\vec{r} = (3\hat{i} - 15\hat{j} + 9\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 5\hat{k})$, and

$\vec{r} = (-\hat{i} + \hat{j} + 9\hat{k}) + \mu(2\hat{i} + \hat{j} - 3\hat{k})$

do not intersect.

11. Show that the lines

$\vec{r} = (2\hat{i} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$ and $\vec{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$

intersect.

Also, find their point of intersection.

12. Show that the lines

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\vec{r} = (4\hat{i} + \hat{j}) + \mu(5\hat{i} + 2\hat{j} + \hat{k})$

intersect.

Also, find their point of intersection.

13. Find the shortest distance between the lines L_1 and L_2 whose vector equations are **[CBSE 2014]**

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \text{ and}$$

$$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 6\hat{k}).$$

HINT: The given lines are parallel.

14. Find the distance between the parallel lines L_1 and L_2 whose vector equations are

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$, and $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$.

15. Find the vector equation of a line passing through the point (2, 3, 2) and parallel to the line $\vec{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$. Also, find the distance between these lines.

HINT: The given line is $L_1 : \vec{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$.

The required line is $L_2 : \vec{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(2\hat{i} - 3\hat{j} + 6\hat{k})$.

Now, find the distance between the parallel lines L_1 and L_2.

16. Write the vector equation of each of the following lines and hence determine the distance between them: **[CBSE 2010]**

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6} \text{ and } \frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}.$$

HINT: The given lines are $L_1 : \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

$L_2 : \vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 6\hat{k})$.

Now, find the distance between the parallel lines L_1 and L_2.

17. Write the vector equations of the following lines and hence find the shortest distancebetween them: **[CBSE 2010C]**

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-5}{5}.$$

Find the shortest distance between the lines given below:

18. $\frac{x-1}{-1} = \frac{y+2}{1} = \frac{z-3}{-2}$ and $\frac{x-1}{1} = \frac{y+1}{2} = \frac{z+1}{-2}$.

19. $\frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2}$ and $\frac{x-23}{-6} = \frac{y-19}{-4} = \frac{z-25}{3}$.

HINT: Change the given equations in vector form.

ANSWERS (EXERCISE 27D)

1. $\frac{10}{\sqrt{59}}$ units **2.** $\sqrt{62}$ units **3.** $\frac{3\sqrt{19}}{19}$ units **4.** $\frac{3\sqrt{2}}{2}$ units

5. $\frac{14\sqrt{241}}{241}$ units **6.** $\sqrt{38}$ units **7.** $\sqrt{35}$ units **8.** $\frac{5\sqrt{2}}{2}$ units

9. SD $= \dfrac{\sqrt{14}}{14} \neq 0$, the given lines do not intersect **11.** (2, 6, 3)

12. (−1, −1, −1) **13.** $\dfrac{\sqrt{293}}{7}$ units **14.** $\sqrt{26}$ units

15. $L_2 : \vec{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(2\hat{i} - 3\hat{j} + 6\hat{k})$, $\dfrac{\sqrt{580}}{7}$ units

16. $\left.\begin{aligned} L_1 : \vec{r} &= (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \\ L_2 : \vec{r} &= (3\hat{i} + 3\hat{j} - 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 6\hat{k}) \end{aligned}\right\}$, SD $= \dfrac{\sqrt{293}}{7}$ units

17. $\left.\begin{aligned} L_1 : \vec{r} &= (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \\ L_2 : \vec{r} &= (2\hat{i} + 3\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} + 5\hat{k}) \end{aligned}\right\}$, SD $= \dfrac{\sqrt{6}}{6}$ units

18. $\dfrac{8\sqrt{29}}{29}$ units **19.** 26 units

Shortest Distance between Two Skew Lines in the Cartesian Form

The shortest distance between the skew lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \text{ and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

is given by

$$\text{SD} = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{D}},$$

where $D = \{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2\}$.

CONDITION FOR TWO GIVEN LINES TO INTERSECT Let L_1 and L_2 be the given lines whose equations are

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \text{ and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}.$$

1. L_1 and L_2 intersect $\Leftrightarrow$ SD between them is 0

$$\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

2. L_1 and L_2 do not intersect $\Leftrightarrow$ they are skew lines

$$\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the shortest distance between the lines*

$$\frac{x+3}{-4}=\frac{y-6}{3}=\frac{z}{2} \text{ and } \frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}.$$

SOLUTION Comparing the given equations with

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}, \text{ we get}$$

$$(x_1=-3,\ y_1=6,\ z_1=0),\ (x_2=-2,\ y_2=0,\ z_2=7),$$

$$(a_1=-4,\ b_1=3,\ c_1=2) \text{ and } (a_2=-4,\ b_2=1,\ c_2=1).$$

$$\text{Now, } D=(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2$$

$$=(-4+12)^2+(3-2)^2+(-8+4)^2$$

$$=(64+1+16)=81.$$

$$\therefore \quad \text{SD}=\frac{1}{\sqrt{D}}\cdot\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$=\frac{1}{\sqrt{81}}\cdot\begin{vmatrix} -2+3 & 0-6 & 7-0 \\ -4 & 3 & 2 \\ -4 & 1 & 1 \end{vmatrix}=\frac{1}{9}\cdot\begin{vmatrix} 1 & -6 & 7 \\ -4 & 3 & 2 \\ -4 & 1 & 1 \end{vmatrix}$$

$$=\frac{1}{9}\cdot\{1\cdot(3-2)+6\cdot(-4+8)+7\cdot(-4+12)\}$$

$$=\frac{81}{9}=9 \text{ units.}$$

Hence, the shortest distance between the given lines is 9 units.

EXAMPLE 2 *Find the length and the equations of the line of shortest distance between the lines*

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}.$$ **[CBSE 2008, '14]**

SOLUTION The equations of the given lines are

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}=\lambda \text{ (say)} \qquad \ldots \text{(i)}$$

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}=\mu \text{ (say).} \qquad \ldots \text{(ii)}$$

Any point on (i) is $P(7\lambda-1, -6\lambda-1, \lambda-1)$.

Any point on (ii) is $Q(\mu+3, -2\mu+5, \mu+7)$.

The direction ratios of PQ are $(\mu-7\lambda+4, -2\mu+6\lambda+6, \mu-\lambda+8)$.

Now, PQ will be the shortest distance between (i) and (ii) only when PQ is perpendicular to both (i) and (ii).

$$\therefore \quad \begin{cases} 7(\mu - 7\lambda + 4) - 6(-2\mu + 6\lambda + 6) + 1\cdot(\mu - \lambda + 8) = 0 \\ \text{and} \\ 1\cdot(\mu - 7\lambda + 4) - 2(-2\mu + 6\lambda + 6) + 1\cdot(\mu - \lambda + 8) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 20\mu - 86\lambda = 0 \\ 6\mu - 20\lambda = 0 \end{cases} \Rightarrow \begin{cases} 10\mu - 43\lambda \\ 3\mu - 10\lambda \end{cases} \Rightarrow \lambda = 0 \text{ and } \mu = 0.$$

$\therefore$ PQ will be the line of shortest distance when $\lambda = 0$ and $\mu = 0$.

Putting $\lambda = 0$ and $\mu = 0$, we get the points $P(-1, -1, -1)$ and $Q(3, 5, 7)$.

$$\therefore \quad \text{SD} = |PQ| = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2}$$

$$= \sqrt{(4)^2 + (6)^2 + (8)^2} = \sqrt{16 + 36 + 64}$$

$$= \sqrt{116} = 2\sqrt{29} \text{ units.}$$

Clearly, the equation of the line of shortest distance is the equation of line PQ given by

$$\frac{x-3}{3+1} = \frac{y-5}{5+1} = \frac{z-7}{7+1}$$

$$\Leftrightarrow \quad \frac{x-3}{4} = \frac{y-5}{6} = \frac{z-7}{8}$$

$$\Leftrightarrow \quad \frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}.$$

Hence, the equation of the line of shortest distance is

$$\frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}.$$

EXAMPLE 3 *Find the length and the equations of the line of shortest distance between the lines*

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

SOLUTION The equations of the given lines are

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} = \lambda \text{ (say)} \qquad \text{... (i)}$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} = \mu \text{ (say)} \qquad \text{... (iii)}$$

Any point on the line (i) is $P(3\lambda + 8, -16\lambda - 9, 7\lambda + 10)$.

Any point on the line (ii) is $Q(3\mu + 15, 8\mu + 29, -5\mu + 5)$.

The direction ratios of PQ are

$$(3\mu - 3\lambda + 7, 8\mu + 16\lambda + 38, -5\mu - 7\lambda - 5).$$

Now, PQ will be the shortest distance between (i) and (ii) only when PQ is perpendicular to each one of (i) and (ii).

$$\therefore \quad \begin{cases} 3(3\mu - 3\lambda + 7) - 16(8\mu + 16\lambda + 38) + 7(-5\mu - 7\lambda - 5) = 0 \\ \text{and} \\ 3(3\mu - 3\lambda + 7) + 8(8\mu + 16\lambda + 38) - 5(-5\mu - 7\lambda - 5) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 314\lambda + 154\mu + 622 = 0 \\ 154\lambda + 98\mu + 350 = 0 \end{cases} \Rightarrow \begin{cases} 157\lambda + 77\mu + 311 = 0 & \ldots(iii) \\ 77\lambda + 49\mu + 175 = 0. & \ldots(iv) \end{cases}$$

On multiplying (iii) by 7 and (iv) by 11 and subtracting, we get

$$(1099\lambda - 847\lambda) + (2177 - 1925) = 0 \Rightarrow 252\lambda = -252 \Rightarrow \lambda = -1.$$

Putting $\lambda = -1$ in (iv), we get

$$49\mu + (175 - 77) = 0 \Rightarrow 49\mu + 98 = 0 \Rightarrow 49\mu = -98 \Rightarrow \mu = -2.$$

Now, $\lambda = -1$ gives $P(5, 7, 3)$ and $\mu = -2$ gives $Q(9, 13, 15)$.

$$\therefore \quad \text{SD} = |PQ| = \sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2}$$

$$= \sqrt{16 + 36 + 144} = \sqrt{196} = 14 \text{ units.}$$

Equation of the line of shortest distance is the equation of PQ, given by

$$\frac{x-5}{9-5} = \frac{y-7}{13-7} = \frac{z-3}{15-3}$$

$$\Rightarrow \quad \frac{x-5}{4} = \frac{y-7}{6} = \frac{z-3}{12} \Rightarrow \frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}.$$

Hence, the equation of the line of shortest distance is

$$\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}.$$

EXAMPLE 4 *Show that the lines*

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-4}{5} = \frac{y-1}{2} = z$$

intersect each other. Find their point of intersection. **[CBSE 2004]**

SOLUTION The equations of the given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda \text{ (say)} \qquad \ldots (i)$$

$$\frac{x-4}{5} = \frac{y-1}{2} = \frac{z-0}{1} = \mu. \qquad \ldots (ii)$$

Any point on the line (i) is $P(2\lambda + 1,\ 3\lambda + 2,\ 4\lambda + 3)$.

Any point on the line (ii) is $Q(5\mu + 4,\ 2\mu + 1,\ \mu)$.

If the lines (i) and (ii) intesect then P and Q must coincide for some particular values of λ and μ.

This gives

$$2\lambda + 1 = 5\mu + 4,\ 3\lambda + 2 = 2\mu + 1 \text{ and } 4\lambda + 3 = \mu$$

$$\Rightarrow \begin{cases} 2\lambda - 5\mu = 3 & \ldots (i) \\ 3\lambda - 2\mu = -1 & \ldots (ii) \\ 4\lambda - \mu = -3. & \ldots (iii) \end{cases}$$

On solving (i) and (ii), we get $\lambda = -1$ and $\mu = -1$.

These values of λ and μ also satisfy (iii).

Hence, the given lines intersect.

Putting $\lambda = -1$, we get $P(-1, -1, -1)$.

Note that putting $\mu = -1$, we get $Q(-1, -1, -1)$.

Hence, the point of intersection of the given lines is $(-1, -1, -1)$.

EXAMPLE 5 *Show that the lines*

$$\frac{x-1}{2} = \frac{y+1}{3} = z \text{ and } \frac{x+1}{5} = \frac{y-2}{2}, z = 2$$

do not intersect each other. **[CBSE 2010]**

SOLUTION The equations of the given lines are

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-0}{1} = \lambda \text{ (say)} \quad \text{... (i)}$$

$$\frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0} = \mu \text{ (say)} \quad \text{... (ii)}$$

Any point on the line (i) is $P(2\lambda + 1, 3\lambda - 1, \lambda)$.

Any point on the line (ii) is $Q(5\mu - 1, \mu + 2, 2)$.

If the lines (i) and (ii) intersect, then P and Q must coincide for some particular values of λ and μ.

This gives

$$2\lambda + 1 = 5\mu - 1,\ 3\lambda - 1 = \mu + 2 \text{ and } \lambda = 2$$

$$\Rightarrow \begin{cases} 2\lambda - 5\mu = -2 & \text{... (iii)} \\ 3\lambda - \mu = 3 & \text{... (iv)} \\ \lambda = -3. & \text{... (v)} \end{cases}$$

Putting $\lambda = 2$ in (iv), we get $\mu = 3$.

Clearly, $\lambda = 2$ and $\mu = 3$ do not satisfy (iii).

Hence, the given lines do not intersect each other.

EXERCISE 27E

Find the length and the equations of the line of shortest distance between the lines given by

1. $\frac{x-3}{3} = \frac{y-8}{-1} = z - 3$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.
2. $\frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}$ and $\frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}$.
3. $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$ and $\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$.

4. $\frac{x-6}{3}=\frac{y-7}{-1}=\frac{z-4}{1}$ and $\frac{x}{-3}=\frac{y+9}{2}=\frac{z-2}{4}$.

5. Show that the lines
$$\frac{x}{1}=\frac{y-2}{2}=\frac{z+3}{3} \text{ and } \frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4}$$
intersect and find their point of intersection.

6. Show that the lines
$$\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5} \text{ and } \frac{x-2}{2}=\frac{y-1}{3}=\frac{z+1}{-2}$$
do not intersect each other.

ANSWERS (EXERCISE 27E)

1. $3\sqrt{30}$ units, $\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$ **2.** $\sqrt{35}$ units, $\frac{x-4}{-1}=\frac{y-2}{-3}=\frac{z+3}{5}$

3. $4\sqrt{3}$ units, $x=y=z$ **4.** $3\sqrt{30}$ units, $\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$

5. (2, 6, 3)

EXERCISE 27F

Very-Short-Answer Questions

1. If a line has direction ratios 2, –1, –2 then what are its direction cosines? **[CBSE 2012]**

2. Find the direction cosines of the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$. **[CBSE 2013C]**

3. If the equations of a line are $\frac{3-x}{-3}=\frac{y+2}{-2}=\frac{z+2}{6}$, find the direction cosines of a line parallel to the given line. **[CBSE 2012]**

4. Write the equations of a line parallel to the line $\frac{x-2}{-3}=\frac{y+3}{2}=\frac{z+5}{6}$ and passing through the point (1, –2, 3). **[CBSE 2009C]**

5. Find the Cartesian equations of the line which passes through the point (–2, 4, –5) and which is parallel to the line $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$. **[CBSE 2013]**

6. Write the vector equation of a line whose Cartesian equations are $\frac{x-5}{3}=\frac{y+4}{7}=\frac{6-z}{2}$. **[CBSE 2010, '11]**

7. The Cartesian equations of a line are $\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$. Write the vector equation of the line. **[CBSE 2014]**

8. Write the vector equation of a line passing through the point (1, –1, 2) and parallel to the line whose equations are $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z+1}{-2}$. **[CBSE 2013C]**

9. If $P(1, 5, 4)$ and $Q(4, 1, -2)$ be two given points, find the direction ratios of PQ. **[CBSE 2008]**

10. The equations of a line are $\frac{4-x}{2}=\frac{y+3}{2}=\frac{z+2}{1}$. Find the direction cosines of a line parallel to this line. **[CBSE 2013]**

11. The Cartesian equations of a line are $\frac{x-1}{2}=\frac{y+2}{3}=\frac{5-z}{1}$. Find its vector equation.

12. Find the vector equation of a line passing through the point (1, 2, 3) and parallel to the vector $(3\hat{i}+2\hat{j}-2\hat{k})$.

13. The vector equation of a line is $\vec{r}=(2\hat{i}+\hat{j}-4\hat{k})+\lambda(\hat{i}-\hat{j}-\hat{k})$. Find its Cartesian equation.

14. Find the Cartesian equation of a line which passes through the point (–2, 4, –5) and which is parallel to the line $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$.

15. Find the Cartesian equation of a line which passes through the point having position vector $(2\hat{i}-\hat{j}+4\hat{k})$ and is in the direction of the vector $(\hat{i}+2\hat{j}-\hat{k})$.

16. Find the angle between the lines $\vec{r}=(2\hat{i}-5\hat{j}+\hat{k})+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$ and $\vec{r}=(7\hat{i}-6\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$.

17. Find the angle between the lines $\frac{x+3}{3}=\frac{y-1}{5}=\frac{z+3}{4}$ and $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$.

18. Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are at right angles.

19. The direction ratios of a line are 2, 6, –9. What are its direction cosines?

20. A line makes angles 90°, 135° and 45° with the positive directions of x-axis, y-axis and z-axis respectively. What are the direction cosines of the line?

21. What are the direction cosines of the y-axis?

22. What are the direction cosines of the vector $(2\hat{i}+\hat{j}-2\hat{k})$?

23. What is the angle between the vector $\vec{r}=(4\hat{i}+8\hat{j}+\hat{k})$ and the x-axis?

ANSWERS (EXERCISE 27F)

1. $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$ **2.** $\frac{-2}{7}, \frac{6}{7}, \frac{-3}{7}$ **3.** $\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$

4. $\frac{x-1}{-3} = \frac{y+2}{2} = \frac{z-3}{6}$ **5.** $\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$

6. $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} - 2\hat{k})$

7. $\vec{r} = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k})$

8. $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 2\hat{k})$ **9.** $3, -4, -6$

10. $\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}$ **11.** $\vec{r} = (\hat{i} - 2\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$

12. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$ **13.** $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z+4}{-1}$

14. $\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$ **15.** $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$ **16.** $\cos^{-1}\left(\frac{19}{21}\right)$

17. $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$ **19.** $\frac{2}{11}, \frac{6}{11}, \frac{-9}{11}$ **20.** $0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ **21.** $0, 1, 0$

22. $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$ **23.** $\cos^{-1}\left(\frac{4}{9}\right)$

HINTS TO THE GIVEN QUESTIONS (EXERCISE 27F)

1. D.r.'s of the given line are $2, -1, -2$ and $\sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$.

   $\therefore$ d.c.'s of the given line are $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$.

2. The given equations are $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$.

   $\therefore$ d.r.'s of the given line are $-2, 6, -3$ and $\sqrt{(-2)^2 + 6^2 + (-3)^2} = \sqrt{49} = 7$.

   $\therefore$ d.c.'s of the given line are $\frac{-2}{7}, \frac{6}{7}, \frac{-3}{7}$.

3. The given equations are $\frac{x-3}{3} = \frac{y+2}{-2} = \frac{z+2}{6}$.

   $\therefore$ d.r.'s of this line are $3, -2, 6$.

   $\therefore$ d.r.'s of a line parallel to this line are $3, -2, 6$ and $\sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{49} = 7$.

   $\therefore$ d.c.'s of a line parallel to the given line are $\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$.

4. D.r.'s of the given line are $-3, 2, 6$.

   D.r.'s of a line parallel to the given line are $-3, 2, 6$.

   Required equations of the line are $\frac{x-1}{-3} = \frac{y+2}{2} = \frac{z-3}{6}$.

5. The equations of the given line are $\frac{x+3}{3}=\frac{y-4}{-5}=\frac{z+8}{6}$.

D.r.'s of this line are 3, –5, 6.

D.r.'s of a line parallel to this line are 3, –5, 6.

Required equations of a line through (–2, 4, –5) are $\frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}$.

6. The given equations of the line are $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{-2}$.

Clearly, this line passes through the point (5, –4, 6) and it has d.r.'s 3, 7, –2.

$\therefore$ its vector equation is $\vec{r}=(5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}-2\hat{k})$.

7. The given equations of the line are $\frac{x-3}{-5}=\frac{y-(-4)}{7}=\frac{z-3}{2}$.

Clearly, this line passes through the point (3, –4, 3) and it has d.r.'s –5, 7, 2.

$\therefore$ its vector equation is $\vec{r}=(3\hat{i}-4\hat{j}+3\hat{k})+\lambda(-5\hat{i}+7\hat{j}+2\hat{k})$.

8. Clearly, the requried line passes through the point (1, –1, 2) and its d.r.'s are 1, 2, –2.

Hence, its vector equation is $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\lambda(\hat{i}+2\hat{j}-2\hat{k})$.

9. D.r.'s of PQ are $(4-1),(1-5),(-2-4)$, i.e., 3, –4, –6.

10. The given equations are $\frac{x-4}{-2}=\frac{y+3}{2}=\frac{z+2}{1}$.

The d.r.'s of this line are –2, 2, 1.

The d.r.'s of a line parallel to this line are –2, 2, 1 and $\sqrt{(-2)^2+2^2+1^2}=\sqrt{9}=3$.

The d.c.'s of a line parallel to the given line are $\frac{-2}{3},\frac{2}{3},\frac{1}{3}$.

11. The given equations are $\frac{x-1}{2}=\frac{y-(-2)}{3}=\frac{z-5}{-1}$.

Clearly, this line passes thorugh the point (1, –2, 5) and it has d.r.'s 2, 3, –1.

So, its vector equation is $\vec{r}=(\hat{i}-2\hat{j}+5\hat{k})+\lambda(2\hat{i}+3\hat{j}-\hat{k})$.

12. Clearly, the required equation is $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(3\hat{i}+2\hat{j}-2\hat{k})$.

13. The given line passes through the point (2, 1, –4) and it is parallel to a line whose direction ratios are 1, –1, –1.

So, its Cartesian equation is $\frac{x-2}{1}=\frac{y-1}{-1}=\frac{z+4}{-1}$.

14. Clearly, the line passes through the point (–2, –4, –5) and it has direction ratios 3, 5, 6.

So, its equation is $\frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}$.

15. The required line passes through the point (2, –1, 4) and it has direction ratios 1, 2, –1.

$\therefore$ its equation is $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z-4}{-1}$.

16. The angle between the lines and $\vec{r} = a_1 + \lambda b_1$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$ is given by $\cos\theta = \dfrac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$.

$$\therefore \quad \cos\theta = \frac{|(3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})|}{\{\sqrt{3^2 + 2^2 + 6^2}\}\{\sqrt{1^2 + 2^2 + 2^2}\}} = \frac{(3 + 4 + 12)}{(7 \times 3)} = \frac{19}{21}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{19}{21}\right).$$

17. Here $(a_1) = 3,\ b_1 = 5,\ c_1 = 4)$ and $(a_2 = 1,\ b_2 = 1,\ c_2 = 2)$.

$$\therefore \quad \cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\{\sqrt{a_1^2 + b_1^2 + c_1^2}\}\{\sqrt{a_2^2 + b_2^2 + c_2^2}\}}$$

$$= \frac{|(3 \times 1) + (5 \times 1) + (4 \times 2)|}{\{\sqrt{3^2 + 5^2 + 4^2}\}\{\sqrt{1^2 + 1^2 + 2^2}\}}$$

$$= \frac{16}{(\sqrt{50} \times \sqrt{6})} = \frac{16}{10\sqrt{3}} = \frac{8}{5\sqrt{3}}.$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\right) = \cos^{-1}\left(\frac{8\sqrt{3}}{15}\right).$$

18. Here $(a_1 = 7,\ b_1 = -5,\ c_1 = 1)$ and $(a_2 = 1,\ b_2 = 2,\ c_2 = 3)$.

$$(a_1a_2 + b_1b_2 + c_1c_2) = (7 \times 1) + (-5) \times 2 + (1 \times 3) = 0.$$

Hence, the given lines are at right angles.

19. We have $\sqrt{2^2 + 6^2 + (-9)^2} = \sqrt{121} = 11$.

$$\therefore \quad \text{d.c.'s of the given line are } \frac{2}{11}, \frac{6}{11}, \frac{-9}{11}.$$

20. D.c.'s of the line are $\cos 90°$, $\cos 135°$ and $\cos 45°$, i.e., $\left(0, \dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{2}}\right)$.

21. Clearly, the y-axis makes an angle of 90°, 0°, 90° with the x-axis, y-axis and z-axis respectively.

So, its d.c.'s are $\cos 90°$, $\cos 0°$, $\cos 90°$, i.e., 0, 1, 0.

22. D.r.'s of the given vector are 2, 1, –2 and $\sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$.

$$\therefore \quad \text{d.c.'s of the given vector are } \frac{2}{3}, \frac{1}{3}, \frac{-2}{3}.$$

23. D.r.'s of the given vector are 4, 8, 1 and $\sqrt{4^2 + 8^2 + 1^2} = \sqrt{81} = 9$.

$$\therefore \quad \text{d.c.'s of the given vector are } \frac{4}{9}, \frac{8}{9}, \frac{1}{9}.$$

Let α be the angle between the given vectors and the x-axis.

Then, $\cos\alpha = \dfrac{4}{9} \Rightarrow \alpha = \cos^{-1}\left(\dfrac{4}{9}\right)$.

OBJECTIVE QUESTIONS

Multiple-Choice Questions

1. The direction ratios of two lines are 3, 2, –6 and 1, 2, 2 respectively. The acute angle between these lines is

   (a) $\cos^{-1}\left(\frac{5}{18}\right)$ (b) $\cos^{-1}\left(\frac{3}{20}\right)$

   (c) $\cos^{-1}\left(\frac{5}{21}\right)$ (d) $\cos^{-1}\left(\frac{8}{21}\right)$

2. The direction ratios of two lines are a, b, c and $(b-c), (c-a), (a-b)$ respectively. The angle between these lines is

   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

3. The angle between the lines $\frac{x-2}{2}=\frac{y-1}{7}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{2}=\frac{z-5}{4}$ is

   (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\cos^{-1}\left(\frac{3}{8}\right)$

4. If the lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$ are perpendicular to each other then $k=?$

   (a) $\frac{-5}{7}$ (b) $\frac{5}{7}$ (c) $\frac{10}{7}$ (d) $\frac{-10}{7}$

5. A line passes through the points $A(2, -1, 4)$ and $B(1, 2, -2)$. The equations of the line AB are

   (a) $\frac{x-2}{-1}=\frac{y+1}{2}=\frac{z-4}{-6}$ (b) $\frac{x+2}{-1}=\frac{y+1}{2}=\frac{z-4}{6}$

   (c) $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z-4}{6}$ (d) none of these

6. The angle between the lines $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x-5}{4}=\frac{y-2}{1}=\frac{z-3}{8}$ is

   (a) $\cos^{-1}\left(\frac{3}{4}\right)$ (b) $\cos^{-1}\left(\frac{5}{6}\right)$ (c) $\cos^{-1}\left(\frac{2}{3}\right)$ (d) $\frac{\pi}{3}$

7. The angle between the lines
   $\vec{r}=(3\hat{i}+\hat{j}-2\hat{k})+\lambda(\hat{i}-\hat{j}-2\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j}-5\hat{k})+\mu(3\hat{i}-5\hat{j}-4\hat{k})$ is

   (a) $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$ (b) $\cos^{-1}\left(\frac{6\sqrt{2}}{5}\right)$ (c) $\cos^{-1}\left(\frac{5\sqrt{3}}{8}\right)$ (d) $\cos^{-1}\left(\frac{5\sqrt{2}}{6}\right)$

8. A line is perpendicular to two lines having direction ratios 1, –2, –2 and 0, 2, 1. The direction cosines of the line are

   (a) $\frac{-2}{3}, \frac{1}{3}, \frac{2}{3}$ (b) $\frac{2}{3}, \frac{1}{3}, \frac{-1}{3}$ (c) $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$ (d) none of these

9. A line passes through the point $A(5, -2. 4)$ and it is parallel to the vector $(2\hat{i} - \hat{j} + 3\hat{k})$. The vector equation of the line is

(a) $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(5\hat{i} - 2\hat{j} + 4\hat{k})$

(b) $\vec{r} = (5\hat{i} - 2\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$

(c) $\vec{r} \cdot (5\hat{i} - 2\hat{j} + 4\hat{k}) = \sqrt{14}$

(d) none of these

10. The Cartesian equations of a line are $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-5}{-1}$. Its vector equation is

(a) $\vec{r} = (-\hat{i} + 2\hat{j} - 5\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$

(b) $\vec{r} = (2\hat{i} + 3\hat{j} - \hat{k}) + \lambda(\hat{i} - 2\hat{j} + 5\hat{k})$

(c) $\vec{r} = (\hat{i} - 2\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 4\hat{k})$

(d) none of these

11. A line passes through the point $A(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$. The vector equation of the line is

(a) $\vec{r} = (-3\hat{i} + 4\hat{j} - 8\hat{k}) + \lambda(-2\hat{i} + 4\hat{j} - 5\hat{k})$

(b) $\vec{r} = (-2\hat{i} + 4\hat{j} - 5\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 6\hat{k})$

(c) $\vec{r} = (3\hat{i} + 5\hat{j} + 6\hat{k}) + \lambda(-2\hat{i} + 4\hat{j} - 5\hat{k})$

(d) none of these

12. The coordinates of the point where the line through the points $A(5, 1, 6)$ and $B(3, 4, 1)$ crosses the yz-plane is

(a) $(0, 17, -13)$ (b) $\left(0, \frac{-17}{2}, \frac{13}{2}\right)$

(c) $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$ (d) none of these

13. The vector equation of the x-axis is given by

(a) $\vec{r} = \hat{i}$ (b) $\vec{r} = \hat{j} + \hat{k}$ (c) $\vec{r} = \lambda\hat{i}$ (d) none of these

14. The Cartesian equations of a line are $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-3}{-2}$. What is its vector equation?

(a) $\vec{r} = (2\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$

(b) $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 2\hat{k})$

(c) $\vec{r} = (2\hat{i} + 3\hat{j} - 2\hat{k})$

(d) none of these

15. The angle between two lines having direction ratios 1, 1, 2 and $(\sqrt{3}-1)$, $(-\sqrt{3}-1)$, 4 is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

16. The straight line $\frac{x-2}{3}=\frac{y-3}{1}=\frac{z+1}{0}$ is

(a) parallel to the x-axis (b) parallel to the y-axis
(c) parallel to the z-axis (d) perpendicular to the z-axis

17. If a line makes angles α, β and γ with the x-axis, y-axis and z-axis respectively then $(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)$ =?

(a) 1 (b) 3 (c) 2 (d) $\frac{3}{2}$

18. If (a_1, b_1, c_1) and (a_2, b_2, c_2) be the direction ratios of two parallel lines then

(a) $a_1=a_2, b_1=b_2, c_1=c_2$ (b) $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$
(c) $a_1^2+b_1^2+c_1^2=a_2^2+b_2^2+c_2^2$ (d) $a_1a_2+b_1b_2+c_1c_2=0$

19. If the points $A(-1, 3, 2)$, $B(-4, 2, -2)$ and $C(5, 5, \lambda)$ are collinear then the value of λ is

(a) 5 (b) 7 (c) 8 (d) 10

ANSWERS (OBJECTIVE QUESTIONS)

1. (c) **2.** (b) **3.** (c) **4.** (d) **5.** (a) **6.** (c) **7.** (a) **8.** (c)
9. (b) **10.** (c) **11.** (b) **12.** (c) **13.** (c) **14.** (b) **15.** (c) **16.** (d)
17. (c) **18.** (b) **19.** (d)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. (c) $\cos\theta=\frac{|(3\times1)+(2\times2)+(-6)\times2|}{\{\sqrt{3^2+2^2+(-6)^2}\}\{\sqrt{1^2+2^2+2^2}\}}=\frac{|-5|}{(\sqrt{49}\times\sqrt{49})}=\frac{5}{(7\times3)}=\frac{5}{21}$

$=\frac{1-51}{(\sqrt{49}\times\sqrt{9})}=\frac{5}{(7\times3)}=\frac{5}{21}$

$\Rightarrow \quad \theta=\cos^{-1}\left(\frac{5}{21}\right)$.

2. (b) $\cos\theta=\frac{a(b-c)+b(c-a)+c(a-b)}{\{\sqrt{a^2+b^2+c^2}\}\{\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}\}}=0 \Rightarrow \theta=\frac{\pi}{2}$.

3. (c) The d.r.'s of these lines are 2, 7, –3 and –1, 2, 4.

$\therefore \quad \cos\theta=\frac{|2\times(-1)+7\times2+(-3)\times4|}{\{\sqrt{2^2+7^2+(-3)^2}\}\{\sqrt{(-1)^2+2^2+4^2}\}}=0 \Rightarrow \theta=\frac{\pi}{2}$.

4. (d) Since the given lines are perpendicular to each other, we have

$$(-3)(3k)+(2k\times 1)+2\times(-5)=0 \Rightarrow 7k=-10 \Rightarrow k=\frac{-10}{7}.$$

5. (a) The given line passes through the point $A(2,-1,4)$ and its direction ratios are $(1-2),(1+1),(-2-4)$, i.e., $-1, 2, -6$.

$\therefore$ required equations of the line are $\frac{x-2}{-1}=\frac{y+1}{2}=\frac{x-4}{-6}$.

6. (c) The d.r.'s of the given lines are 2, 2, 1 and 4, 1, 8.

$$\therefore \quad \cos\theta=\frac{|(2\times 4)+(2\times 1)+(1\times 8)|}{\{\sqrt{2^2+2^2+1^2}\}\{\sqrt{4^2+1^2+8^2}\}}=\frac{18}{(\sqrt{9}\times\sqrt{81})}=\frac{18}{(3\times 9)}=\frac{2}{3}$$

$$\Rightarrow \quad \theta=\cos^{-1}\left(\frac{2}{3}\right).$$

7. (a) The given lines are $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\lambda\vec{b_2}$.

$$\therefore \quad \cos\theta=\frac{|\vec{b_1}\cdot\vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}=\frac{|(\hat{i}-\hat{j}-2\hat{k})\cdot(3\hat{i}-5\hat{j}-4\hat{k})|}{\{\sqrt{1^2+(-1)^2+(-2)^2}\}\{\sqrt{3^2+(-5)^2+(-4)^2}\}}$$

$$=\frac{(3+5+8)}{(\sqrt{6}\times\sqrt{50})}=\frac{16}{10\sqrt{3}}=\frac{8}{5}\times\frac{\sqrt{3}}{3}=\frac{8}{15}\sqrt{3}$$

$$\Rightarrow \quad \theta=\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right).$$

8. (c) Let the direction ratios of the required line be a, b, c. Then,

$$a-2b-2c=0 \qquad \text{... (i)}$$

$$0a+2b+c=0. \qquad \text{... (ii)}$$

On solving (i) and (ii) by cross multiplication, we get

$$\frac{a}{(-2+4)}=\frac{b}{(0-1)}=\frac{c}{(2-0)} \Rightarrow \frac{a}{2}=\frac{b}{-1}=\frac{c}{2}.$$

$\therefore$ d.r.'s of the required line are 2, –1, 2 and $\sqrt{2^2+(-1)^2+2^2}=\sqrt{9}=3$.

$\therefore$ d.c.'s of the required line are $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$.

9. (b) Clearly, the required vector equation of the line is

$$\vec{r}=(5\hat{i}-2\hat{j}+4\hat{k})+\lambda(2\hat{i}-\hat{j}+3\hat{k}).$$

10. (c) The line passes through the point (1, –2, 5) and it has direction ratios 2, 3, –1.

So, its vector equation is $\vec{r}=(\hat{i}-2\hat{j}+5\hat{k})+\lambda(2\hat{i}+3\hat{j}-\hat{k})$.

11. (b) The required line passes through the point $(-2\hat{i}+4\hat{j}+5\hat{k})$ and it is parallel to the vector $(3\hat{i}+5\hat{j}+6\hat{k})$.

Hence, the required equation of the line is

$$\vec{r}=(-2\hat{i}+4\hat{j}-5\hat{k})+\lambda(3\hat{i}+5\hat{j}+6\hat{k}).$$

12. (c) Equation of the line AB is

$$\frac{x-5}{5-3}=\frac{y-1}{1-4}=\frac{z-6}{6-1} \Rightarrow \frac{x-5}{2}=\frac{y-1}{-3}=\frac{z-6}{5}=\lambda.$$

Any point on this line is $(2\lambda+5, -3\lambda+1, 5\lambda+6)$.

If this point lies on the yz-plane, we have $2\lambda+5=0 \Rightarrow \lambda=\frac{-5}{2}$.

$\therefore$ the required point is $\left(0, \frac{15}{2}+1, \frac{-25}{2}+6\right)$, i.e., $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$.

13. (c) Note that the Cartesian equation of x-axis is $\frac{x-0}{1}=\frac{y-0}{0}=\frac{z-0}{0}$ and its vector equation is $\vec{r}=\lambda\hat{i}$.

14. (b) The line passes through the point $(2\hat{i}-\hat{j}+3\hat{k})$ and it is parallel to the vector $(2\hat{i}+3\hat{j}-2\hat{k})$.

So, its vector equation is $\vec{r}=(2\hat{i}-\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}-2\hat{k})$.

15. (c) $\cos\theta=\frac{1\times(\sqrt{3}-1)+1\times(-\sqrt{3}-1)+2\times4}{\{\sqrt{1^2+1^2+2^2}\}\{\sqrt{(\sqrt{3}-1)^2+(-\sqrt{3}-1)^2+4^2}\}}$

$$=\frac{(\sqrt{3}-1-\sqrt{3}-1+8)}{(\sqrt{6}\times\sqrt{24})}=\frac{\sqrt{6}}{\sqrt{24}}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}.$$

16. (c) If a line is perpendicular to the z-axis then $\cos\gamma=\cos\frac{\pi}{2}=0$.

So, the given line is perpendicular to the z-axis.

17. (c) We have $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.

$\therefore \quad (1-\sin^2\alpha)+(1-\sin^2\beta)+(1-\sin^2\gamma)=1$

$\Rightarrow \quad \sin^2\alpha+\sin^2\beta+\sin^2\gamma=2.$

18. (b) If the given lines are parallel, then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

19. (c) Equations of the line AB are

$$\frac{x+1}{-4+1}=\frac{y-3}{2-3}=\frac{z-2}{-2-2} \Rightarrow \frac{x+1}{-3}=\frac{y-3}{-1}=\frac{z-2}{-4}$$

$$\Rightarrow \quad \frac{x+1}{3}=\frac{y-3}{1}=\frac{z-2}{4}.$$

If the point $C(5, 5, \lambda)$ lies on AB, we have

$$\frac{5+1}{3}=\frac{5-3}{1}=\frac{\lambda-2}{4} \Rightarrow \frac{\lambda-2}{4}=2 \Rightarrow \lambda=(8+2)=10.$$